Lectures on Integrable Models in Field/String Theory

Lecture 1

Introduction

Hamilton's principle. Noether's theorem. Gauge symmetry. Noether's second theorem. Hamiltonian description. Faddeev-Jackiw formalism.

Lecture 2

Symplectic geometry

The Hamiltonian dynamics. Definitions and notations. Useful formulas and identities. Hamiltonian vector fields. Darboux theorem. Symplectic structure on TQ, T^*Q and on the space of solutions. Moment map. Co-cycles of Lie algebras and central extensions.

Lecture 3

The $SL(2,\mathbb{R})$ group

The $sl(2,\mathbb{R})$ algebra. The Killing form. The exponential map: $sl(2,\mathbb{R}) \to SL(2,\mathbb{R})$. The adjoint representation. Coordinates on $SL(2,\mathbb{R})$. Functions, vector fields, 1-forms and the metric on the $SL(2,\mathbb{R})$ group manifold.

Lecture 4

Particle dynamics on symmetric spaces

The Liouville model. The dynamics of a particle in $SU(2)$ (classical and quantum theories). Dynamics of a relativistic particle in $SL(2,\mathbb{R})$. Particle dynamics in AdS space. The dynamics of a massless particle.

Lecture 5

Gauging and Hamiltonian reduction

Gauging of Noether symmetries. Singular Lagrangian. First order formalism. Reductions of differential forms. Examples: Mechanical model of QED. Gauging of the particle dynamics on $SU(2)$ and $SL(2,\mathbb{R})$ group manifolds.

Lecture 6

The method of co-adjoint orbits

Co-adjoint representation of Lie groups. Co-adjoint orbits. Symplectic forms and Hamiltonian vector fields on co-adjoint orbits. Geometric quantization. Choice of polarization. Irreducible representations.

Lecture 7

Geometric quantization and coherent states

Symmetries and coherent states. Examples: Weyl group, $SL(2,\mathbb{R})$ and $SU(2)$ coherent states. Symbol calculus. Moyal quantization. Coherent state formalism and geometric quantization.

Lecture 8

The Lagrangian formulation of $SL(2,\mathbb{R})$ WZW theory

σ-models in 2-dimensions. The $SL(2,\mathbb{R})$ target space. 2-forms on $SL(2,\mathbb{R})$ group manifold and the $SL(2,\mathbb{R})$ WZW Lagrangian. The general solution and global symmetries. The $SU(2)$ WZW Lagrangian and the WZ term. Symmetries and integration of dynamical equations.

Lecture 9

The Symplectic structure of 2d free-field theory

Free field theory on a cylinder and a strip. Canonical form. Chiral fields and the chiral symplectic form. The Poisson brackets algebra of chiral fields. 'Vertex functions' and their algebra. The energy momentum tensor and the conformal symmetry.

Lecture 10

The Hamiltonian formulation of WZW theory

Canonical structure of WZW theory. The chiral symplectic form. The Poisson brackets algebra of chiral WZ fields. Kac-Moody algebra. The Sugawara energy momentum tensor. $SU(2)$ and $SL(2,\mathbb{R})$ WZW models.

Lecture 11

Gauging of WZW theory

Vector and axial gauging of $SL(2,\mathbb{R})$ WZW theory. $U(1)$ gauging and $SL(2,\mathbb{R})/U(1)$ black hole model. $\mathbb{R}^1$ gauging. Nilpotent gauging and Liouville theory. Hamiltonian reduction and free-field parametrization.

Lecture 12

Canonical quantization of 2d CFT

Canonical quantization of free-field theory. 2d conformal symmetry and Virasoro algebra. Vertex operators and their algebra. Canonical map to Liouville theory. Construction of Liouville vertex operators and calculation of the reflection amplitude.

Lecture 13

Geometric quantization of infinite dimensional symmetries

The co-adjoint orbits of Virasoro group. Symplectic structure and Poisson brackets. Transformation to free-field variables. Coherent states of infinite dimensional translation group and 2d conformal group. Transition amplitudes between the coherent states. Kac-Moody group.

Lecture 14

String dynamics in Minkowsky space

Dynamical equations and gauge fixing. Integration of equations of motion. Light-cone gauge quantization. Covariant quantization and critical dimension. Polyakov method and non-critical strings. Static gauge quantization.

Lecture 15

The AdS/CFT correspondence

String dynamics in AdS and $AdS \times S$ spaces. Lax pair representation of the dynamical equations. Integration by the Pohlmeyer method. Light-cone quantization of $AdS_5 \times S^5$ string dynamics. Static gauge quantization of AdS strings. $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Main ideas of the correspondence. Integrable structures of the dual theories.

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Lecture 1

Introduction

Hamilton's principle

Let us consider a D-dimensional spacetime X with coordinates x^μ ($\mu = 0, 1, 2, ..., D-1$) and a set of fields $\Phi^k(x)$ ($\kappa = 1, 2, ..., N$) on X, i.e. $x = (x^0, x^1, \ldots, x^{D-1})$. Similarly, $\Phi = (\Phi^1, \Phi^2, \ldots, \Phi^N)$ will denote the set of fields $\Phi^k(x)$ and $\partial\Phi$ the set of their first order derivatives $\partial_\mu \Phi^k(x)$. The action of the system is then introduced by

$$S[\Phi, \Omega] = \int_\Omega \mathrm{d}^D x \; \mathcal{L}(\Phi(x), \partial\Phi(x), x) \;, \tag{1.1}$$

where Ω is a domain in X and $\mathcal{L}$ is a Lagrangian. The equations of motion

$$\frac{\partial \mathcal{L}}{\partial \Phi^k} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^k)} \right) = 0 \tag{1.2}$$

are obtained by the Hamilton's principle, which states that the action functional (1.1) is stationary on the solutions of the equation of motion with fixed values of the fields $\Phi^k(x)$ at the boundary of Ω.

Note that point mechanics corresponds to $D = 1$, $x^0 = t$ and $\Phi^k(x) = q^k(t)$. In this case, the Euler-Lagrange equations (1.2) are equivalent to

$$W_{kl}(q, \dot{q}, t)\, \ddot{q}^{\,l} = V_k(q, \dot{q}, t) \;, \tag{1.3}$$

with

$$W_{kl} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^k \partial \dot{q}^l} \;, \qquad V_k = \frac{\partial \mathcal{L}}{\partial q^k} - \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^k \partial q^l}\, \dot{q}^l \;. \tag{1.4}$$

The matrix W_{kl} is called Hessian. Equation (1.3) takes a Newtonian form

$$\ddot{q}^{\,k} = F^k(q, \dot{q}, t) \tag{1.5}$$

for a non-degenerated Hessian and the corresponding Lagrangian is called regular.

For a regular Lagrangian the map from the velocities $\dot{q}^k$ to the canonical momenta

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}^k} \tag{1.6}$$

is invertible, $\dot{q}^k = v^k(p, q, t)$, and the system of second order equations (1.3) is equivalent to

$$\dot{q}^k = \frac{\partial H}{\partial p_k} \;, \qquad \dot{p}_k = -\frac{\partial H}{\partial q^k} \;, \tag{1.7}$$

where H is the canonical Hamiltonian

$$H = \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^k}\, \dot{q}^k - \mathcal{L} \right) |_{\dot{q}^k = v^k(p,q,t)} \;. \tag{1.8}$$

The solutions of the Hamilton equations (1.7) are stationary points of the action

$$S = \int_{t_i}^{t_f} dt[p_k\, \dot{q}^k - H(p, q, t)] \ . \tag{1.9}$$

The coordinates and momenta (q^k, p_k) are canonical variables on the phase space and an observable A is a function of the canonical variables and time $A = A(p, q, t)$.

The Poisson brackets

$$\{A,\, B\} = \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q^k} - \frac{\partial A}{\partial q^k}\frac{\partial B}{\partial p_k} \tag{1.10}$$

define the Lie algebra of observables and allow to write the Hamilton equations (1.7) in a symmetric form

$$\dot{q}^k = \{H, q^k\} \ , \qquad \dot{p}_k = \{H, p_k\} \ . \tag{1.11}$$

The dynamical equation of an observable $A(p, q, t)$ then becomes

$$\dot{A} = \partial_t A + \{H,\, A\} \ . \tag{1.12}$$

The canonical momenta in field theory are introduced similarly to (1.6)

$$\Pi_k(t,\, \vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^k} \ , \tag{1.13}$$

where the following notations are used $t = x^0$, $\vec{x} = (x^1, \ldots, x^{D-1})$ and $\dot{\Phi}^k = \partial_0 \Phi^k$. The Euler-lagrange equations (1.2) are usually equivalent to the Hamilton equations

$$\dot{\Phi}^k(\vec{x}) = \frac{\delta H}{\delta \Pi_k(\vec{x})} \ , \qquad \dot{\Pi}_k(\vec{x}) = -\frac{\delta H}{\delta \Phi^k(\vec{x})} \ , \tag{1.14}$$

with

$$H = \int \mathrm{d}\vec{x} \left(\Pi_k(\vec{x})\dot{\Phi}^k(\vec{x}) - \mathcal{L}(\vec{x})\right) \ . \tag{1.15}$$

Here, the integration is performed over $(D-1)$ spatial coordinates, at fixed time t, and certain boundary conditions for the functions $\Pi_k(\vec{x})$ and $\Phi^k(\vec{x})$ are assumed. Boundary conditions should provide consistency of the Hamilton equations (1.14) and their equivalence to the Euler-Lagrange equations (1.2). Some standard boundary conditions are discussed below for a 2-dimensional filed theory.

Observables are functionals of the canonical variables $\big(\Pi_k(\vec{x}),\, \Phi^k(\vec{x})\big)$ and the Poisson brackets are defined by

$$\{A,\, B\} = \int \mathrm{d}\,\vec{x} \left(\frac{\delta A}{\delta \Pi_k(\vec{x})}\frac{\delta B}{\delta \Phi^k(\vec{x})} - \frac{\delta A}{\delta \Phi^k(\vec{x})}\frac{\delta B}{\delta \Pi_k(\vec{x})}\right) \ . \tag{1.16}$$

This gives to the Hamilton equations the same form as in (1.11)

$$\dot{\Phi}^k(\vec{x}) = \{H, \Phi^k(\vec{x})\} \ , \qquad \dot{\Pi}_k = \{H, \Pi_k(\vec{x})\} \ , \tag{1.17}$$

only now the canonical variables are labeled by the index k and the spatial coordinate $\vec{x}$.

As an example we consider 2-dimensional (2d) field theory with the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\,\partial_\mu \Phi(x)\partial^\mu \Phi(x) \ , \tag{1.18}$$

where $\partial^\mu = \eta^{\mu\nu}\partial_\nu$ and $\eta^{\mu\nu} = \mathrm{diag}(-1,1)$ is the metric tensor of 2d Minkowski space.

From (1.2) we obtain the wave equation $(\partial_0^2 - \partial_1^2)\Phi = 0$, which we write in the form

$$\ddot{\Phi}(\tau,\sigma) - \Phi''(\tau,\sigma) = 0 \ , \tag{1.19}$$

where dot and prime denote time ($\tau = x^0$) and space ($\sigma = x^1$) derivatives, respectively.

By (1.13) and (1.18), we get $\Pi(\sigma) = \dot{\Phi}(\sigma)$ and the Hamiltonian (1.15) becomes

$$H = \frac{1}{2}\int \mathrm{d}\sigma\,\left[\Pi^2(\sigma) + \Phi'^2(\sigma)\right] \ , \tag{1.20}$$

with some integration domain in σ. The case $-\infty < \sigma < \infty$ corresponds to 2d Minkowski space. Two other standard cases are the field theory on the cylinder $0 \le \sigma \le 2\pi$ and on the strip $0 \le \sigma \le \pi$. For these cases we consider the following boundary conditions:

$$1.\quad -\infty < \sigma < \infty \ , \quad \Phi'(\sigma) \to 0 \ , \quad \Pi(\sigma) \to 0 \ , \quad s \to \pm\infty \ ; \tag{1.21}$$

$$2.\quad 0 \le \sigma \le 2\pi \ , \quad \Phi(0) = \Phi(2\pi), \quad \Pi(0) = \Pi(2\pi) \ ; \tag{1.22}$$

$$3.\quad 0 \le \sigma \le \pi \ , \quad \Phi'(0) = \Phi'(\pi) = 0 \ , \quad \Pi'(0) = \Pi'(\pi) = 0 \ . \tag{1.23}$$

The case (1.21) is called the open boundary conditions and it usually assumes vanishing of fields and their derivatives at the spatial infinity. The second case (1.22) corresponds to the periodic boundary conditions and the third case to the Neuman boundary conditions. These tree cases lead to the Hamilton equations

$$\dot{\Phi}(\sigma) = \Pi(\sigma) \ , \qquad \dot{\Pi}(\sigma) = \Phi''(\sigma) \ , \tag{1.24}$$

which are equivalent to (1.19). As it was mentioned above, boundary conditions in field theory have to provide consistency of the Hamilton equations.

Noether's theorems

Noether's theorem relates symmetry transformations to conservation laws.

We consider so called point transformations of coordinates and fields

$$x^\mu \mapsto \tilde{x}^\mu = y^\mu(x) \ , \qquad \Phi^k(x) \mapsto \tilde{\Phi}^k(\tilde{x}) = \Psi^k(\Phi, x) \ . \tag{1.25}$$

Here, each y^μ is a function of D coordinates x^ν, each Ψ^k is a function of N fields Φ^l and D coordinates x^ν. The map (1.25) is assumed invertible.

The Lagrangian for the fields $\tilde{\Phi}^k(\tilde{x})$ is obtained by the condition $\tilde{S}[\tilde{\Phi}, \tilde{\Omega}] = S[\Phi, \Omega]$, which provides

$$\tilde{\mathcal{L}}(\tilde{\Phi}(\tilde{x}), \tilde{\partial}\tilde{\Phi}(\tilde{x}), \tilde{x}) = \frac{\partial x}{\partial \tilde{x}}\,\mathcal{L}(\Phi(x), \partial\Phi(x), x) \ , \tag{1.26}$$

where $\frac{\partial x}{\partial \tilde{x}} = \det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right)$ is the Jacobian of the coordinate transformation.

Introducing the Euler derivative

$$E_k = \frac{\partial \mathcal{L}}{\partial \Phi^k} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^k)} \right) \ , \tag{1.27}$$

from (1.26) be obtain

$$\tilde{E}_k(\tilde{x}) = \frac{\partial x}{\partial \tilde{x}} \frac{\partial \Phi^l}{\partial \tilde{\Phi}^k} E_l(x) \ . \tag{1.28}$$

Thus, the fields $\tilde{\Phi}(\tilde{x})$ also satisfy the Euler-Lagrange equations with the Lagrangian (1.26). However, it does not mean that the Euler-Lagrange equations are form-invariant.

The transformation (1.25) is called a symmetry transformation if the new Lagrangian $\tilde{\mathcal{L}}$ has the same functional form as the initial one, i.e.

$$\mathcal{L}(\tilde{\Phi}(\tilde{x}), \tilde{\partial}\tilde{\Phi}(\tilde{x}), \tilde{x}) = \frac{\partial x}{\partial \tilde{x}} \mathcal{L}(\Phi(x), \partial\Phi(x), x) \ . \tag{1.29}$$

In this case the form of the equations of motion for the fields $\tilde{\Phi}^k(\tilde{x})$ and $\Phi^k(x)$ is the same.

Now we consider a special type of symmetry transformations related to a Lie group G.

Let G be a group with M real parameters ϵ^α ($\alpha = 1, 2, \ldots, M$) which acts both on the spacetime X and on the space of fields Φ. Assuming that $\epsilon = 0$ corresponds to the unit element of the group, we can write the infinitesimal action of the group in the form

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \epsilon^\alpha \, v^\mu_\alpha(x) + \mathcal{O}(\epsilon^2) \ , \qquad \Phi^k \mapsto \tilde{\Phi}^k = \Phi^k + \epsilon^\alpha \, V^k_\alpha(\Phi) + \mathcal{O}(\epsilon^2) \ . \tag{1.30}$$

Note that the operators

$$\hat{v}_a = v^\mu_\alpha \partial_\mu \ , \qquad \hat{V}_\alpha = V^k_\alpha \partial_k \ , \tag{1.31}$$

satisfy the commutation relations

$$[\hat{v}_\alpha, \, \hat{v}_\beta] = C_{\alpha\beta}{}^\gamma \hat{v}_\gamma \ , \qquad [\hat{V}_\alpha, \, \hat{V}_\beta] = C_{\alpha\beta}{}^\gamma \hat{V}_\gamma \ , \tag{1.32}$$

where $C_{\alpha\beta}{}^\gamma$ are the structure constants of G.

If (1.30) is a symmetry transformation then the first Noether's theorem states that

$$\partial_\mu J^\mu_\alpha + E_\kappa V^k_\alpha(\Phi) = 0 \ , \tag{1.33}$$

where E_κ are Euler derivatives (1.27) and

$$J^\mu_\alpha = v^\mu_a \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^k)} \left(V^k_\alpha - v^\nu_\alpha \, \partial_\nu \Phi^k \right) \ . \tag{1.34}$$

Hence,

$$\partial_\mu J^\mu_\alpha = 0 \ , \tag{1.35}$$

due to the Euler-Lagrange equations, and one finds the conserved observables

$$Q_\alpha = \int \mathrm{d}\vec{x} \, J^0_\alpha(t, \vec{x}) \ . \tag{1.36}$$

These Q_α are called Noether charges and $J^\mu_\alpha(x)$ conserved currents.

A symmetric action, that leads to conservation laws, can be constructed by some invariants. However, such constructions usually involve non physical degrees of freedom and create gauge symmetries. Note that in gauge transformations group parameters are arbitrary functions of spacetime coordinates. For illustration we consider two simple examples.
1. A relativistic particle in Minkowski space is described by the action

$$S = -m \int \mathrm{d}\tau \sqrt{-\dot{q}^2} \tag{1.37}$$

where $\dot{q}^2 = \eta_{\mu\nu}\dot{q}^\mu \dot{q}^n$ and $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the metric tensor of Minkowski space. This action is proportional to the length of the particle worldline and therefor it is invariant under the Poincare transformations

$$q^\mu \mapsto \tilde{q}^\mu = \Lambda^\mu{}_\nu q^\nu + a^\mu \ , \tag{1.38}$$

which define the isometry group of the Minkowski space. However, at the same time, the length is invariant under the reparametrizations $t \mapsto f(t)$, with arbitrary monotonic $f(t)$. These are the gauge transformations mentioned above.

The Noether charges for the symmetry transformations (1.38) read

$$P^\mu = \frac{m\dot{q}^\mu}{\sqrt{-\dot{q}^2}} \ , \qquad M_{\mu\nu} = P_\mu q_\nu - P_\nu q_\mu \ , \tag{1.39}$$

and their conservation is provided by the equations of motion obtain from (1.37)

$$\frac{m}{\sqrt{-\dot{q}^{1/2}}} \left[\ddot{q}_\mu - \frac{\dot{q}_\mu(\dot{q}\ddot{q})}{\dot{q}^2} \right] = 0 \ . \tag{1.40}$$

2. The action for the electro-magnetic field is given by

$$S = -\frac{1}{4} \int \mathrm{d}^4x F_{\mu\nu}F^{\mu\nu} \tag{1.41}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This action leads to the Poincare covariant field equations

$$\partial_\mu F^{\mu\nu} = 0 \ , \tag{1.42}$$

and at the same time it is invariant under the gauge transformations

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \lambda(x) \ . \tag{1.43}$$

A similar situation is in Yang-Mills theory, in the standard model and in string theory. Note that general relativity is also a gauge invariant theory.

An important structure of gauge theories is provided by the second Noether's theorem. It states that the Euler derivatives of a gauge theory are linearly related to each other.

One usually considers two different type of gauge transformations. The first corresponds to spacetime diffeomorphisms and the second to gauge transformations only of fields, as it was illustrated in the above mentioned two examples. We present these cases separately.

First we consider gauge transformations of coordinates and the corresponding transformations of fields written in the following infinitesimal form

$$\tilde{x}^\mu = x^\mu + \epsilon^\mu(x) + \mathcal{O}(\epsilon^2) \ , \qquad \tilde{\Phi}^k(\tilde{x}) = \Phi^k(x) + \partial_\nu \epsilon^\mu(x)\, V^{k,\nu}_\mu(\Phi) + \mathcal{O}(\epsilon^2) \ . \tag{1.44}$$

If the action of the system is invariant under these gauge transformations, then the linear relations between the Euler derivatives take the form

$$\partial_\mu \Phi^k\, E_k - \partial_\nu \left(V^{k,\nu}_\mu\, E_k \right) = 0 \ . \tag{1.45}$$

In the second type gauge symmetries one transforms only fields

$$\Phi^k(x) \mapsto \tilde{\Phi}^k(x) = \Phi^k(x) + \epsilon^\alpha(x)\, V^\kappa_\alpha(\Phi) + \partial_\nu \epsilon^\alpha(x)\, V^{k,\nu}_\alpha(\Phi) + \mathcal{O}(\epsilon^2) \ . \tag{1.46}$$

and the linear relations between the Euler derivatives become

$$V^k_\alpha\, E_k - \partial_\nu \left(V^{k,\nu}_\alpha\, E_k \right) = 0 \ . \tag{1.47}$$

Note that the number of linear relations in both cases coincide with the number of gauge group parameters and in the first case it is given by the spacetime dimension.

Linear relations between the Euler derivatives implies a degeneracy of the Hessian. The Lagrangian of a gauge theory, therefore, is singular and the standard Hamiltonian description fails.

Hamiltonian treatment of singular systems was introduced by Dirac and later on there were many attempts to develop the method due to its importance for the fundamental theories. Here we present the Faddeev-Jackiw formalism which appeared quite effective for the description of gauge theories and other constraint systems as well.

Faddeev-Jackiw formalism

Let us consider a mechanical system with coordinates $q = (q^1, q^2, \ldots, q^N)$ and the action[1]

$$S[q] = \int dt\, L(q, \dot{q}) \ . \tag{1.48}$$

One can introduce $2N$ additional coordinates $p = (p_1, p_2, \ldots, p_n)$, $v = (v^1, v^2, \ldots v^n)$ and the new action

$$\tilde{S}[q, p, v] = \int \mathrm{d}t \left[p_k(\dot{q}^k - v^k) + L(q, v) \right] \ , \tag{1.49}$$

where $L(q, v)$ is the same function as in (1.48). The variation of (1.49) with respect to p_k provides $\dot{q}^k = v^k$ and their insertion in (1.49) leads to the initial action (1.48).

Thus, these two actions provide the same dynamics for the coordinates q^k.

One can also consider the variation of (1.49) with respect to v^k, which yields

$$p_k = \frac{\partial L(q, v)}{\partial v^k} \ . \tag{1.50}$$

[1]Since the equations of motion and other local properties of a system are independent on boundary conditions, the integration domain sometimes is not important and it is not indicated in the action.

For a regular Lagrangian these equations allow to express the variables v^k in terms of (p, q) and one ends up with the Hamiltonian system (1.9).

However, if the Lagrangian is singular, one can solve only a part of the variables v^k. Suppose, these are $(v^1, v^2, \dots v^{N-M})$. Without loss of generality one can assume that after solving the first $N - M$ variables $(v^1, v^2, \dots v^{N-M})$ the dependance on the rest variables v^a $(a = N - M + 1, \dots, N)$ is linear and the action (1.49) takes the following form

$$\tilde{S} = \int \mathrm{d}t \left[p_k \dot{q}^k - H(p, q) - v^a \phi_a(p, q)\right] \ . \tag{1.51}$$

Then varying v^a lead to the constraints

$$\phi_a(p, q) = 0 \ . \tag{1.52}$$

These constraints define $(2N - M)$-dimensional surface in (p, q) space. The point on the surface can be parameterized by $2N - M$ coordinates ξ^α $(\alpha = 1, 2, \dots, 2N - M)$

$$q^k = q^k(\xi) \ , \qquad p_k = p_k(\xi) \ , \tag{1.53}$$

and the action (1.51) in the new coordinates becomes

$$\tilde{S}[\xi] = \int \mathrm{d}t \left[\theta_\alpha(\xi)\dot{\xi}^\alpha - H(\xi)\right] \ , \tag{1.54}$$

where $\theta_a(\xi) = p_k(\xi)\partial_a q^k(x)$. For further analysis one has to calculate the rank r of the antisymmetric matrix

$$\omega_{\alpha\beta}(\xi) = \partial_\alpha \theta_\beta(\xi) - \partial_\beta \theta_\alpha(\xi) \ , \tag{1.55}$$

which is even $r = 2n$. According to the Darboux's theorem there exists a transformation to new variables (P_i, Q^i, η^γ), where $i = 1, 2, \dots, n$ and $\gamma = 1, 2, \dots, 2N - M - 2n$, such that

$$\theta_\alpha \mathrm{d}\xi^\alpha = P_i \mathrm{d}Q^i + \mathrm{d}F(P, Q, \eta) \ . \tag{1.56}$$

Neglecting the derivative term $\mathrm{d}F(P, Q, \eta)$, one gets from (1.54)

$$\tilde{S} = \int \mathrm{d}t \left[P_i \dot{Q}^i - H(P, Q, \eta)\right] \ . \tag{1.57}$$

Varying now η^γ, one finds algebraic equations for η's

$$\partial_\gamma H(P, Q, \eta) = 0 \ . \tag{1.58}$$

Either all η's can be excluded from these equations or one gets new constraints on (P, Q) and the reduction procedure has to be repeated again. In this way one finally obtains an ordinary Hamiltonian system (1.9).

This reduction procedure for a gauge invariant theory usually stops at the level (1.57) and at the level (1.51) the functions $\phi_a(p, q)$ and H provide the Poisson brackets algebra

$$\{\phi_a, \phi_b\} = C_{ab}{}^c \phi_c \ , \qquad \{H, \phi_a\} = 0 \ , \tag{1.59}$$

where $C_{ab}{}^c$ are the structure constants of the gauge group.

Exercises

1. Derive the Euler-Lagrange equations (1.2) from the Hamilton's principle.

2. Check the equivalence of the equations of motion (1.3) and (1.7).

3. Derive the Hamilton's equation (1.7) by the variation of (1.9) and specify the boundary conditions.

4. Check the canonical Poisson brackets

$$\{\Pi_k(\vec{x}),\, \Pi_l(\vec{y})\} = 0 = \{\Phi^k(\vec{x}),\, \Phi^l(\vec{y})\}~, \qquad \{\Pi_k(\vec{x}),\, \Phi^l(\vec{y})\} = \delta_k^l\, \delta(\vec{x}-\vec{y})~. \tag{1.60}$$

5. Derive the Hamilton equations (1.24), using the Hamiltonian (1.20) and the boundary conditions (1.21)-(1.23).

6. Check the relation (1.28) for the Euler derivatives.

7. Derive the commutation relations (1.28).

8. Let g_ϵ be a one parameter group which acts on the time coordinate t by the rule

$$g_{\epsilon_2}(g_{\epsilon_1}(t)) = g_{\epsilon_1+\epsilon_2}(t)~, \tag{1.61}$$

and its infinitesimal form is given by

$$g_\epsilon(t) = t + \epsilon\, v(t) + \mathcal{O}(\epsilon^2)~. \tag{1.62}$$

Find the global form of $g_\epsilon(t)$ in terms of the function $v(t)$.

9. Let us consider a set of transformations (1.62) labeled by integer n

$$g_{\epsilon^n}(t) = t + \epsilon^n\, v_n(t) + \mathcal{O}(\epsilon^2)~, \tag{1.63}$$

with $v_n(t) = t^{n+1}$. Show that the vector fields $\hat{v}_n$ form a Lie algebra and calculate its structure constants. Show that v_{-1}, v_0 and v_1 form a subalgebra and calculate the corresponding global transformations of t.

10. Let us consider a mechanical system with Noether charges

$$Q_\alpha = v_\alpha(t)\mathcal{L}(q,\dot{q},t) + \frac{\partial \mathcal{L}}{\partial \dot{q}^k}\left(V_\alpha^k(q) - v_\alpha\, \dot{q}^k\right)~. \tag{1.64}$$

Check whether the Poisson brackets of Q_α form the algebra (1.32). Investigate the same problem in field theory.

11. Derive the equations of motion (1.37) and check the linear relation (1.45).

12. Derive the equations of motion (1.41) and check the linear relation (1.47).

13. Check the equations (1.45) and (1.47).

14. Apply the Faddeev-Jackiw reduction to the relativistic particle action (1.37).

15. Apply the Faddeev-Jackiw reduction to the action of electro-magnetic field (1.41).

16. Let us consider a Lagrangian with second order derivatives

$$S[\Phi] = \int_\Omega \mathrm{d}T \; \mathcal{L}(\Phi^k, \partial_\mu \Phi^k, \partial^2_{\mu\nu} \Phi^k) \; . \tag{1.65}$$

Formulate the stationary action principle and derive the equations of motion

$$\frac{\partial \mathcal{L}}{\partial \Phi^k} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^k)} \right) + \partial^2_{\mu\nu} \left(\frac{\partial \mathcal{L}}{\partial(\partial^2_{\mu\nu} \Phi^k)} \right) = 0 \; . \tag{1.66}$$

17. Consider a symmetry transformation for the action with second order derivatives and construct the corresponding conserved current J^μ_α.

18. Describe symmetries of the model (1.18) and find the corresponding conserved currents.

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Lecture 2

Symplectic Geometry

1. The Hamiltonian dynamics

The Hamilton equations

$$\dot{q}^a = \frac{\partial H}{\partial p_a} \ , \qquad \dot{p}_a = -\frac{\partial H}{\partial q^a} \qquad\qquad (a = 1, ..., N) \tag{2.1}$$

can be written in the form

$$\dot{\eta}^m = \frac{1}{2}\, \omega_0^{mn}\, \frac{\partial H}{\partial \eta^n} \ , \tag{2.2}$$

where η^n $(n = 1, ..., 2N)$ combines the phase-space coordinates

$$\eta^1 = p_1, \, ... \, , \eta^N = p_N \ , \qquad \eta^{N+1} = q^1, \, ... \, , \eta^{2N} = q^N \tag{2.3}$$

and ω_0^{mn} is the $2N \times 2N$ antisymmetric matrix

$$\omega_0^{mn} = 2 \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \ . \tag{2.4}$$

The coefficient 1/2 in (2.2) and the corresponding normalization of the matrix ω_0 is chosen just for further convenience. Eq. (2.2) in arbitrary coordinates $\eta \mapsto \xi$ becomes

$$\dot{\xi}^m = \frac{1}{2}\, \omega^{mn}\, \partial_n H \ , \tag{2.5}$$

with

$$\omega^{mn} = \frac{\partial \xi^m}{\partial \eta^k}\, \omega_0^{kl}\, \frac{\partial \xi^n}{\partial \eta^l} \qquad \text{and} \qquad \partial_n \equiv \frac{\partial}{\partial \xi^n} \ . \tag{2.6}$$

Since the Jacobian of the transformation $\eta \mapsto \xi$ is non-zero, the matrix ω^{mn} is non-degenerated and its inverse ω_{mn} provides $\omega^{ml}\,\omega_{ln} = \delta^m{}_n$. The matrixes ω_{mn} and ω^{mn} transform as covariant and contravariant 2-tensors, respectively. They remain antisymmetric, but, in general, they are coordinate dependent. If $\omega^{mn} = \omega_0^{mn}$ the transformation $\eta \mapsto \xi$ is canonical. In this case the initial canonical form of Hamilton equations (2.1) is preserved.

Calculating the canonical Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial p_a}\, \frac{\partial g}{\partial q^a} - \frac{\partial f}{\partial q^a}\, \frac{\partial g}{\partial p_a} \qquad\qquad (a = 1, ..., N) \tag{2.7}$$

in the coordinates ξ^n, one finds

$$\{f, g\} = \frac{1}{2}\, \omega^{mn}\, \partial_n f\, \partial_m g \ . \tag{2.8}$$

Then, the Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \ , \tag{2.9}$$

implies the condition

$$\omega^{kl}\, \partial_k\, \omega^{mn} + \omega^{km}\, \partial_k\, \omega^{nl} + \omega^{kn}\, \partial_k\, \omega^{lm} = 0 \ , \tag{2.10}$$

which for the inverse matrix ω_{mn} leads to

$$\partial_l\, \omega_{mn} + \partial_m\, \omega_{nl} + \partial_n\, \omega_{lm} = 0 \ . \tag{2.11}$$

Exercise 1-5.

2. Definitions and notations from differential geometry

Before introducing the notions of symplectic geometry we recall basic definitions, notations and some useful formulas form differential geometry (see also the appendix of the lecture 3).

- M-dimensional manifold $\mathcal{M}$ locally looks like a domain of $\mathbb{R}^M$.
- ξ^n $(n = 1, ..., M)$ are local coordinates in a domain of $\mathcal{M}$.
- Smooth functions $C^\infty(\mathcal{M})$; $\quad f \in C^\infty(\mathcal{M})$, $\quad f : \mathcal{M} \mapsto \mathbb{R}$.
- Vector fields $V(\mathcal{M})$; $\quad V \in V(\mathcal{M})$, $\quad V = V^n(\xi)\,\partial_n$.
- 1-forms $\Omega^1(\mathcal{M})$; $\quad \Theta \in \Omega^1(\mathcal{M})$, $\quad \Theta = \Theta_n(\xi)\,d\xi^n$.
- The action of 1-forms on vector fields $\Theta(V) = \Theta_n\,V^n$ is equivalent to the rule

$$dx^m(\partial_n) = \delta^m\, n\ . \tag{2.12}$$

- p-forms $\Omega^p(\mathcal{M})$; $\quad \Lambda \in \Omega^p(\mathcal{M})$, $\quad \Lambda = \Lambda_{kl...n}\ d\xi^k \wedge d\xi^l \wedge \cdots \wedge d\xi^n$.
- The action of Λ on p vector fields $\Lambda(V_1, ..., V_p) = \Lambda_{kl...n}\,V_1^k\,V_2^l ... V_p^n$ is provided by

$$\Lambda(\partial_k, \partial_l, ..., \partial_n) = \Lambda_{kl...n}\ . \tag{2.13}$$

- The anti-symmetrization of tensors is given by

$$T_{[kl...n]} = \frac{1}{p!}\sum_\sigma \mathrm{sing}(\sigma)\,T_{\sigma(k)\sigma(l)...\sigma(n)}\ , \tag{2.14}$$

  where the summation is over the permutations σ of p-indices $k, l, ..., n$.

- The exterior product of $\Lambda \in \Omega^p(\mathcal{M})$ and $\Theta \in \Omega^q(\mathcal{M})$; $\quad \Lambda \wedge \Theta \in \Omega^{p+q}(\mathcal{M})$,

$$(\Lambda \wedge \Theta)_{kl...n} = \Lambda_{[kl...}\ \Theta_{m...n]}\ . \tag{2.15}$$

- The differential of a p-form; $\quad d\Lambda \in \Omega^{p+1}(\mathcal{M})$,

$$(d\Lambda)_{kl...n} = \partial_{[k}\,\Lambda_{l...n]}\ . \tag{2.16}$$

- The Lie derivative of a p-form; $\mathcal{L}_V(\Lambda) \in \Omega^p(\mathcal{M})$,

$$\mathcal{L}_V(\Lambda)_{kl...n} = V^m\,\partial_m\,\Lambda_{kl...n} + p\Lambda_{m[l...n}\,\partial_{k]}\,V^m\ . \tag{2.17}$$

- The contraction of V with Λ; $\quad V\rfloor\Lambda \in \Omega^{p-1}(\mathcal{M})$,

$$(V\rfloor\Lambda)_{l...n} = p\,V^m\Lambda_{ml...n}\ . \tag{2.18}$$

- The commutator (the Lie bracket) of two vector fields V and W; $[V, W] \in V(\mathcal{M})$,

$$[V, W]^n = V^m\,\partial_m\,W^n - W^m\,\partial_m\,V^n\ . \tag{2.19}$$

Useful formulas for $\quad \Lambda \in \Omega^p(\mathcal{M})$, $\quad \Theta \in \Omega^q(\mathcal{M})$:

$$d(d\Lambda) = 0 \ , \tag{2.20}$$

$$\Lambda \wedge \Theta = (-1)^{pq}\, \Theta \wedge \Lambda \ , \tag{2.21}$$

$$d(\Lambda \wedge \Theta) = d\Lambda \wedge \Theta + (-1)^p\, \Lambda \wedge d\Theta \ , \tag{2.22}$$

$$V \rfloor (\Lambda \wedge \Theta) = (V \rfloor \Lambda) \wedge \Theta + (-1)^p\, \Lambda \wedge (V \rfloor \Theta) \ , \tag{2.23}$$

$$\mathcal{L}_V \Lambda = V \rfloor d\Lambda + d\,(V \rfloor \Lambda) \ , \tag{2.24}$$

$$\mathcal{L}_V (d\Lambda) = d\,(\mathcal{L}_V\, \Lambda) \ , \tag{2.25}$$

$$\mathcal{L}_V\,(W \rfloor \Lambda) = [V, W] \rfloor \Lambda + W \rfloor \mathcal{L}_V\, \Lambda \ , \tag{2.26}$$

$$(\mathcal{L}_V\, \mathcal{L}_W\ - \mathcal{L}_W\, \mathcal{L}_V)\ \Lambda = \mathcal{L}_{[V,W]}\, \Lambda \ . \tag{2.27}$$

- A form Λ is called closed if $d\Lambda = 0$, and it is called exact if $\Lambda = d\Theta$.
- Due to (2.20), an exact form is closed, but a closed form is not always exact.

Examples with 1- and 2-forms $(\lambda = \lambda_n\, d\xi^n, \quad \theta = \theta_n\, d\xi^n, \quad \omega = \omega_{mn}\, d\xi^m \wedge d\xi^n)$:

$$(\lambda \wedge \theta)_{mn} = \frac{1}{2}\,(\lambda_m \theta_n - \lambda_n \theta_m) \ , \tag{2.28}$$

$$(d\theta)_{mn} = \frac{1}{2}\,(\partial_m \theta_n - \partial_n \theta_m) \ , \tag{2.29}$$

$$(\mathcal{L}_V \theta)_n = V^m\, \partial_m \theta_n + \theta_m\, \partial_n\, V^m \ , \tag{2.30}$$

$$V \rfloor \theta = V^n\, \theta_n \ , \tag{2.31}$$

$$(\omega \wedge \theta)_{mnl} = \frac{1}{3}\ (\omega_{mn}\, \theta_l\ + \omega_{nl}\, \theta_m + \omega_{lm}\, \theta_n) \ , \tag{2.32}$$

$$(d\omega)_{lmn} = \frac{1}{3}\ (\partial_l\, \omega_{mn} + \partial_m\, \omega_{nl} + \partial_n\, \omega_{lm}) \ , \tag{2.33}$$

$$(\mathcal{L}_V \omega)_{mn} = V^l\, \partial_l \omega_{mn} + \omega_{ln}\, \partial_m\, V^l - \omega_{lm}\, \partial_n\, V^l \ , \tag{2.34}$$

$$(V \rfloor \omega)_n = 2V^m\, \omega_{mn} \ . \tag{2.35}$$

Exercise 6-8.

3. Symplectic manifold, Hamiltonian vector fields and Darboux theorem

A 2-form ω is called non-degenerated if the equation for a vector field V

$$V\rfloor\omega = \theta \tag{2.36}$$

has an unique solution for any 1-form θ. Writing ω in local coordinates

$$\omega = \omega_{mn}\, d\xi^m \wedge d\xi^n \ , \tag{2.37}$$

by (2.35) one finds the equation

$$2\, V^m \omega_{mn} = \theta_n \ , \tag{2.38}$$

and the non-degeneracy condition is equivalent to $\det \omega_{mn} \neq 0$. Thus, for a non-degenerated 2-form there exists the inverse matrix ω^{mn} with

$$\omega^{ml}\, \omega_{ln} = \delta^m{}_n \ . \tag{2.39}$$

Since ω_{mn} is anti-symmetric, it can be non-degenerated only if the dimension of $\mathcal{M}$ is even.

A 2-form ω is called symplectic if it is non-degenerated and closed ($d\omega = 0$).

A manifold $\mathcal{M}$ equipped with a symplectic form is a symplectic manifold.

By (2.33), the condition $d\omega = 0$ is just eq. (2.11). To discuss other similarities with the phase space structure, let us consider eq. (2.36) with $\theta = -df$, where f is a function on $\mathcal{M}$

$$df + V_f\rfloor\omega = 0 \ . \tag{2.40}$$

The field V_f is called the Hamiltonian vector field. It is associated with a function f and has the components

$$V_f^n = \frac{1}{2}\, \omega^{nm}\, \partial_m f \ . \tag{2.41}$$

Due to (2.24) and (2.40) the Hamiltonian vector fields preserve the symplectic form

$$\mathcal{L}_{V_f}\, \omega = 0 \ . \tag{2.42}$$

Poisson bracket of two functions f and g is defined by

$$\{\, f, g\,\} = 2\omega(V_f, V_g) \ . \tag{2.43}$$

Using (2.41), it can be written in local coordinates as Eq. (2.8)

$$\{\, f, g\,\} = 2\omega_{mn}\, V_f^m\, V_g^n = \frac{1}{2}\, \omega^{mn}\, \partial_n f\, \partial_m g \ . \tag{2.44}$$

From (2.44) follow other, equivalent to (2.43), expressions

$$\{\, f, g\,\} = V_f\,(g) = -V_g\,(f) \ . \tag{2.45}$$

The Poisson bracket (2.43) is obviously anti-symmetric. As it was mentioned in Section 1, the Jacobi identity (2.9) is equivalent to (2.10). On the other hand, for a non-degenerated ω, equation (2.10) follows from (2.11) (see the exercise 4). It means, that the Jacobi identity is the consequence of the non-degeneracy and the closer of ω.

Finally note that the Hamiltonian vector fields satisfy the commutation relation

$$[\,V_f,\,V_g\,] = V_{\{f,g\}}\;. \tag{2.46}$$

To check this, one can act by the left and right hand sides of (2.46) on a function h. With the help of (2.45) one can verify that the obtained relation is just the Jacobi identity.

Thus, the set of smooth functions on a symplectic manifold form a Lie algebra with respect to the Poisson brackets and the map of this algebra to the Hamiltonian vector fields is a representation of this Lie algebra.

An important example of a symplectic manifold is a standard phase space with canonical coordinates p_a, q^a $(a = 1, ..., N)$ and the canonical symplectic form

$$\omega = dp_a\,\wedge\,dq^a\;. \tag{2.47}$$

In this case the Hamiltonian vector fields are given by

$$V_H = \frac{\partial H}{\partial p_a}\,\frac{\partial}{\partial q^a} - \frac{\partial H}{\partial q^a}\,\frac{\partial}{\partial p_a}\;, \tag{2.48}$$

and the Poisson bracket (2.43) becomes (2.7). As we have seen in Section 1, transformations to arbitrary phase-space coordinates reproduce the formalism of symplectic geometry. It is natural to ask the question: whether a symplectic manifold has the canonical coordinates. The answer is given by Darboux's theorem:

Let $(\mathcal{M},\,\omega)$ be a $2N$-dimensional symplectic manifold and let $m \in \mathcal{M}$. Then there is a neighborhood U of m and a coordinate system (p_a, q^a), $(a = 1, ..., N)$ on U such that $\omega = dp_a\,\wedge\,dq^a$.

In general, canonical (Darboux) coordinates exist only locally. It should be mentioned that their explicit construction sometimes is not easy, even if the canonical structure is global.

In physical applications a symplectic manifold $(\mathcal{M},\,\omega)$ usually arises in a Hamiltonian reduction of a gauge theory to physical (gauge invariant) variables. The obtained symplectic manifold is called the physical phase space.

As we have seen a function on $\mathcal{M}$ plays two roles. It is an observable and at the same time it is a generator of a one parameter group of transformations. A set of functions could generate a Lie group transformations, if their Poisson brackets form a Lie algebra.

Exercise 9-12

4. Symplectic structure on T^*Q, TQ and on the space of solutions

The Lagrangian $L(q, v)$ is a function on TQ, where TQ denotes the tangent space to the configuration space Q. The dynamical trajectories are solutions of the variational equation $\delta S = 0$, with the action

$$S = \int_{t_0}^{t_1} dt\;L(q,\dot q). \tag{2.49}$$

The corresponding Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^a}\right) - \frac{\partial L}{\partial q^a} = 0 \tag{2.50}$$

can be written in the first order form for the $2N$ variables (q^a, v^a)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^a}\right) - \frac{\partial L}{\partial q^a} = 0\;, \qquad \text{and} \qquad \dot q^a = v^a\;. \tag{2.51}$$

The Lagrangian is called regular if

$$\det\left(\frac{\partial^2 L}{\partial v^a \partial v^b}\right) \neq 0 \ . \tag{2.52}$$

In this case the equations

$$p_a = \frac{\partial L}{\partial v^a} \tag{2.53}$$

define velocities v_a as functions of the coordinates q^a and the momenta p_a. Eqs. (2.51), then become equivalent to the Hamilton equations (2.1) with the Hamilton function

$$H = \frac{\partial L}{\partial v^a}\, v^a - L \ . \tag{2.54}$$

The Hamilton equations have the form

$$\dot q^a = V_H^{q^a} \ , \qquad \dot p_a = V_H^{p_a} \ , \tag{2.55}$$

where

$$V_H^{q^a} = \frac{\partial H}{\partial p_a} \ , \qquad V_H^{p_a} = -\frac{\partial H}{\partial q^a} \tag{2.56}$$

are the components of the Hamiltonian vector field (2.48) for the canonical symplectic form (2.47).

Note that the Hamiltonian (2.54), the canonical symplectic form (2.47) and the canonical 1-form $\theta = p_a\, dq^a$ are invariant under the coordinate transformations on Q.

The transformations from the canonical coordinates (q^a, p_a) to (q^a, v^a) can be considered as a change of coordinates on the symplectic manifold. Due to (2.53), the symplectic form in the new coordinates becomes

$$\omega_L = \frac{\partial^2 L}{\partial q^a \partial v^b}\, dq^a \wedge dq^b + \frac{\partial^2 L}{\partial v^a \partial v^b}\, dv^a \wedge dq^b \ . \tag{2.57}$$

Introducing the corresponding Hamiltonian vector field V_H related to the Hamilton function (2.54)

$$V_H \rfloor \omega_L + dH = 0 \ , \tag{2.58}$$

and writing it as

$$V_H = \dot q^a\, \frac{\partial}{\partial q^a} + \dot v^a\, \frac{\partial}{\partial v^a} \ , \tag{2.59}$$

from (2.57) one finds

$$V_H \rfloor \omega_L = \frac{\partial^2 L}{\partial q^a \partial v^b}\, \dot q^a dq^b - \frac{\partial^2 L}{\partial q^b \partial v^a}\, \dot q^a dq^b + \frac{\partial^2 L}{\partial v^a \partial v^b}\, (\dot v^{\,a} dq^b - \dot q^a\, dv^b) \ . \tag{2.60}$$

Now differentiating the Hamiltonian (2.54)

$$dH = \frac{\partial^2 L}{\partial v^a \partial v^b}\, v^a\, dv^b + \frac{\partial^2 L}{\partial v^a \partial q^b}\, v^a\, dq^b - \frac{\partial L}{\partial q^a}\, dq^a \tag{2.61}$$

and inserting (2.60)-(2.61) in eq. (2.58), one indeed obtains the dynamical equations (2.51)

$$\begin{aligned} V_H \rfloor \omega_L + dH = {} & \frac{\partial^2 L}{\partial v^a \partial v^b}\, (v^a - \dot q^a)\, dv^b + \left[\frac{\partial^2 L}{\partial v^a \partial q^b}\, \dot q^b + \frac{\partial^2 L}{\partial v^a \partial v^b}\, \dot v^{\,b} - \frac{\partial L}{\partial q^a}\right] dq^a \\ & + \frac{\partial^2 L}{\partial v^a \partial q^b}\, (v^a - \dot q^a)\, dq^b = 0 \ . \end{aligned} \tag{2.62}$$

The space of solutions (motions) M is the set of functions $q^a = q^a(t)$, which satisfy the dynamical equations (2.50). Note that the trajectories $q^a(t)$ and $q^a(t+s)$, in general, are distinct points of M, even though the two trajectories occupy the same points in Q.

Symmetry properties of integrable systems is usually easier to formulate for M. This space becomes a manifold if we parameterize the solutions $q^a = q^a(t)$ by the initial data $(q^a(t_0), \dot{q}^a(t_0))$. A Hamiltonian field V_H is called complete if the solutions with all admissible initial data can be continued for arbitrary t. When V_H is complete and L is regular, the map

$$\phi(t_0) : M \mapsto TQ : q^a(t) \mapsto (q^a, v^a) = (q^a(t_0), \dot{q}^a(t_0)) \tag{2.63}$$

defines a diffeomorphism and the symplectic form on TQ (given by ω_L) induces the symplectic form on M. There is another, but equivalent, way to define the symplectic structure of M.

Let us consider the following function on M

$$S(t_0, t_1) = \int_{t_0}^{t_1} dt \; L(q, \dot{q}) \;, \tag{2.64}$$

where the action is calculated on the solutions $q = q(t)$. A tangent vector U to M at a solution $q = q(t)$ is a function $u = u(t)$, which satisfies the linearized equation of motion

$$\frac{d}{dt}\left(\frac{\partial^2 L}{\partial v^a \partial v^b}\,\dot{u}^b + \frac{\partial^2 L}{\partial v^a \partial q^b}\,u^b\right) - \frac{\partial^2 L}{\partial q^a \partial v^b}\,\dot{u}^b - \frac{\partial^2 L}{\partial q^a \partial q^b}\,u^b = 0 \;. \tag{2.65}$$

The derivative of the function (2.64) along U is given by

$$U \rfloor dS(t_0, t_1) = \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial q^a}\,u^a + \frac{\partial L}{\partial v^a}\,\dot{u}^a\right) \tag{2.66}$$

$$\left[\frac{\partial L}{\partial v^a}\,u^a\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^a}\right)\right] u^a \;.$$

The last integral vanishes because $q(t)$ is a solution of the equation of motion. Introducing, for each t, a 1-form θ_t on M by

$$U \rfloor \theta_t = u^a(t)\,\frac{\partial L}{\partial v^a} \;, \tag{2.67}$$

we find

$$dS(t_0, t_1) = \theta_{t_1} - \theta_{t_0} \;. \tag{2.68}$$

Here the right-hand side of (2.67) is calculated at $(q, v) = (q(t), \dot{q}(t))$.

The closed 2-form

$$\omega = d\theta_t \tag{2.69}$$

does not depend on t and it coincides with the induced form from TQ.

Comparing the dynamical pictures on M and TQ, one finds a similarity with the quantum case. The dynamics on M is the analog of the Heisenberg picture, while the Schrodinger picture corresponds to the dynamics on TQ.

Exercise 13-14.

Appendix

Moment map

Let $(\mathcal{M}, \omega)$ be a symplectic manifold and let $\mathcal{G}$ be a Lie algebra. The action of $\mathcal{G}$ on $\mathcal{M}$ is a linear map from $A \mapsto V_A$

$$V_{[A,B]} = [V_A, V_B] \ , \tag{2.70}$$

Hamiltonian action

$$h_{[A,B]} = \{h_A, h_B\} \ . \tag{2.71}$$

Moment

$$\mu : m \mapsto f_m, \qquad f_m(A) = h_A(m) \tag{2.72}$$

'Moment' because such a map generalizes the momentum and angular momentum associated with translations and rotations respectively.

Co-cycles of Lie algebras

Let $\mathcal{G}$ be a Lie algebra $\mathcal{G}$ and $\mathcal{G}^*$ its dual. A cocycle on $\mathcal{G}$ is an anti-symmetric bilinear form $\alpha \in \mathcal{G}^* \wedge \mathcal{G}^*$, such that

$$\alpha([A,B],C) + \alpha([B,C],A) + \alpha([C,A],B) = 0 \tag{2.73}$$

for every A, B, C. Any $f \in \mathcal{G}^*$ defines an element δf of $\mathcal{G}^* \wedge \mathcal{G}^*$

$$\delta f(A,B) = \frac{1}{2} f([A,B]) \ . \tag{2.74}$$

Two cocycles are said to be equivalent if they differ by δf. The set of equivalence classes forms a group under addition. It is called the second cohomology group of $\mathcal{G}$ and is denoted by $H^2\mathcal{G}$.

A canonical action $A \mapsto V_A$ of a Lie algebra $\mathcal{G}$ on a symplectic manifold $(\mathcal{M}, \omega)$ determines an element Ω of $H^2\mathcal{G}$. If there is a moment, then $\Omega = 0$.

If $\Omega = 0$ and if each of the vector fields V_A is Hamiltonian, then the action of $\mathcal{G}$ on $\mathcal{M}$ is Hamiltonian.

The usefulness of this statement is that $H^2\mathcal{G} = 0$.

$[\mathcal{G}, \mathcal{G}] = \mathcal{G}$ and $H^2\mathcal{G} = 0$ for any semi-simple Lie algebra.

The Abelian Lie algebras: $[A,B] = 0$ for any A, $B \in \mathcal{G}$. $[\mathcal{G}, \mathcal{G}] = 0$ and $H^2\mathcal{G} = \mathcal{G}^* \wedge \mathcal{G}^*$.

Exercises

1. Prove that if ω^{mn} is given by (2.6), then its inverse is

$$\omega_{mn} = \frac{\partial \eta^k}{\partial \xi^m} \, \omega^0_{kl} \, \frac{\partial \eta^l}{\partial \xi^n} \ , \tag{E.1}$$

where

$$\omega^0_{kl} = \frac{1}{2} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{E.2}$$

inverts (2.4).

2. Derive eq. (2.10) from eq. (2.11).

3. Verify that for $N = 1$, eq. (2.11) is satisfied by any antisymmetric ω.

4. Check that ω_{mn} given by eq. (E.1) satisfies eq. (2.11).

5. Let us consider the transformation $(p, q) \mapsto (\xi^1, \xi^2)$ from the canonical coordinates (p, q) to

$$\xi^1 = p \sqrt{\alpha + \frac{H}{2}} \ , \qquad \xi^2 = q \sqrt{\alpha + \frac{H}{2}} \ . \tag{E.3}$$

Here α is a non-negative parameter and H is the harmonic oscillator hamiltonian

$$H = \frac{p^2 + q^2}{2} \ . \tag{E.4}$$

Check that

$$\{\xi^1, \xi^2\} = H + \alpha = \sqrt{(\xi^1)^2 + (\xi^2)^2 + \alpha^2} \ , \tag{E.5}$$

and that the Poisson brackets of the functions ξ^1, ξ^2 and $\xi^0 = H + \alpha$ form the $sl(2, \mathbb{R})$ algebra.

6. Let us consider on $\mathbb{R}^2 \setminus (0, 0)$ the 1-form

$$\lambda = \frac{x}{x^2 + y^2} \, dy - \frac{y}{x^2 + y^2} \, dx \ . \tag{E.6}$$

Check that λ is closed, but not exact.

7. Derive (2.34) from (2.17).

8. Similarly to the exercise 7, derive the formula for the Lie derivative of a metric tensor.

9. A field V on a symplectic manifold $(\mathcal{M}, \omega)$ is called locally Hamiltonian if it preserves ω

$$\mathcal{L}_V \, \omega = 0 \ . \tag{E.7}$$

Let us consider on $\mathbb{R}^2 \setminus (0, 0)$ the canonical form $\omega = dp \wedge dq$ and the vector field

$$V = \frac{p}{p^2 + q^2} \, \partial_p + \frac{q}{p^2 + q^2} \, \partial_q \ . \tag{E.8}$$

Check that the field (E.8) is locally Hamiltonian, but not Hamiltonian.

10. Check that the commutator of two locally Hamiltonian vector fields V and W is Hamiltonian

$$[V, W] = V_f \ , \tag{E.9}$$

with

$$f = 2\omega(V, W) \ . \tag{E.10}$$

11. Let us consider the 2-form

$$\omega = \frac{1}{\sqrt{\alpha^2 + x^2 + y^2}} \, dx \wedge dy \tag{E.11}$$

on $\mathbb{R}^2$. Here α is a positive parameter. Find a 1-form θ, such that $d\theta = \omega$.

12. Let us consider the 2-form

$$\omega = z \, dx \wedge dy + x \, dy \wedge dz + y \, dz \wedge dx \tag{E.12}$$

on $\mathbb{R}^3$ and its reduction on the unit sphere $x^2 + y^2 + z^2 = 1$. Calculate the reduced 2-form in the spherical coordinates. Check that the reduced form is closed, but not exact. Find the local canonical coordinates for the reduced 2-form.

13. The space of solutions for a free particle is

$$q(t) = x + vt \ , \tag{E.13}$$

where x and v are the coordinate and the velocity at $t = 0$.
Check that this space is invariant under:
• Translations $\quad q(t) \ \mapsto \ q(t) + a + bt$.
• The conformal transformations $\ q(t) \mapsto q(\phi(t)) \, \dot{\phi}(t)^{-1/2}$, with

$$\phi(t) = \frac{\alpha t + \beta}{\gamma t + \delta} \ , \qquad \alpha\delta - \beta\gamma = 1 \ . \tag{E.14}$$

Find the corresponding transformations of the parameters x, v.

14. The space of solutions of the harmonic oscillator is given by

$$q(t) = x \cos t + v \sin t \ , \tag{E.15}$$

where x and v are the coordinate and the velocity at $t = 0$.
Check that this space is invariant under the translations $\quad q(t) \ \mapsto \ q(t) + a \cos t + b \sin t$.

Consider the following transformation of the initial data

$$\begin{pmatrix} x \\ v \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}, \qquad \text{with} \qquad \alpha\delta - \beta\gamma = 1 \ . \tag{E.16}$$

Check that (E.16) is a canonical transformation and find the corresponding transformation of the space of solutions (E.15).

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Lecture 3

The $SL(2,\mathbb{R})$ group

1. The $sl(2,\mathbb{R})$ algebra

A Lie algebra $\mathcal{G}$ is a linear space with a multiplication rule, which is bilinear, antisymmetric and satisfies the Jacobi identity. The product of two vectors $A \in \mathcal{G}$ and $B \in \mathcal{G}$ is called the Lie bracket and it is denoted by $[\,A\,,\,B\,]$. Thus, $[\,A\,,\,B\,] \in \mathcal{G}$ and it satisfies the conditions

$$[\lambda\, A + B\;,\, C\,] = \lambda\,[\,A\,,\,C\,] + [\,B\,,\,C\,]\;, \tag{3.1}$$

$$[\,A\,,\,B\,] = -\,[\,B\,,\,A\,]\;, \tag{3.2}$$

$$[\,[\,A\,,\,B\,]\,,\,C\,] + [\,[\,B\,,\,C\,]\,,\,A\,] + [\,[\,C\,,\,A\,]\,,\,B\,] = 0\;. \tag{3.3}$$

The numbers λ are real for real algebras and to distinguish the real and complex algebras one uses the letters $\mathbb{R}$ and $\mathbb{C}$, respectively.

A representation of $\mathcal{G}$ is a linear map of $\mathcal{G}$ to a space of linear operators $A \mapsto \hat{O}_A$, such that

$$[\,A\,,\,B\,] \mapsto \hat{O}_A\,\hat{O}_B - \hat{O}_B\,\hat{O}_A\;. \tag{3.4}$$

Let us consider the operator ad_A acting on $\mathcal{G}$ by

$$\mathrm{ad}_A(B) = [\,A\,,\,B\,]\;. \tag{3.5}$$

The Jacobi identity (3.3) provides that

$$\mathrm{ad}_{[\,A\,,\,B\,]} = \mathrm{ad}_A\,\mathrm{ad}_B - \mathrm{ad}_B\,\mathrm{ad}_A\;. \tag{3.6}$$

Thus, the map $A \mapsto \mathrm{ad}_A$ defines a representation of $\mathcal{G}$. It is called the adjoint representation.

The $sl(2,\mathbb{R})$ algebra is a remarkable example, which arises in many mathematical and physical constructions. Its elements are 2×2 real traceless matrices and the Lie bracket is the commutator $[\,A\,,\,B\,] = AB - BA$. Since the commutator of two matrixes is traceless, $[\,A\,,\,B\,] \in sl(2,\mathbb{R})$. The conditions (3.1)-(3.3) are obviously fulfilled.

One can use the following basis in $sl(2,\mathbb{R})$

$$T_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \qquad T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad T_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{3.7}$$

Any $A \in sl(2,\mathbb{R})$ can be written as $A = A^n\,T_n$, with real numbers A^n. The basis elements T_n $(n = 0,\,1,\,2)$ satisfy the relations

$$T_m\;T_n = -\eta_{mn}\;I + \epsilon^l{}_{mn}\;T_l\;, \tag{3.8}$$

where I is the unit matrix, $\eta_{mn} = \mathrm{diag}(+,-,-)$ form the metric tensor of $3d$ Minkowski space and ϵ_{mnl} is antisymmetric, with $\epsilon_{012} = 1$. Lover and upper indices are provided by the metric tensor η_{mn} and its inverse η^{mn} $(\eta^{ml}\eta_{ln} = \delta^m_n)$. Due to (3.8), the commutators of T_n are given by

$$[\,T_m,\,T_n\,] = 2\epsilon^l{}_{mn}\;T_l\;. \tag{3.9}$$

The numbers $2\epsilon^l{}_{mn}$ are called the structure constants of the $sl(2,\mathbb{R})$ algebra (see the exercise 1).
Exercises 1-6.

2. The Killing form

Let us introduce a bilinear and symmetric form $\mathcal{B}(A,B)$ on $sl(2,\mathbb{R})$

$$\mathcal{B}(A,B) = \langle\, A\, B\,\rangle \ , \tag{3.10}$$

where the brackets $\langle \cdot \rangle$ denote the following normalized trace of 2×2 matrices

$$\langle\, M\,\rangle = -\,\frac{1}{2}\,\mathrm{Tr}\,(M)\ . \tag{3.11}$$

Deu to (3.8), the normalization (3.11) yields

$$\langle\, T_m\, T_n\,\rangle = \eta_{mn}\ . \tag{3.12}$$

Expanding then A and B in the basis (3.7): $A = A^m\, T_m\,,\ B = B^n\, T_n\,$, we obtain

$$\langle\, A\, B\,\rangle = \eta_{mn}\, A^m\, B^n\ . \tag{3.13}$$

Note that the maps $A^n \mapsto A = A^n\, T_n$ and $A \mapsto A^n = \langle\, A\, T^n\,\rangle$ are inverse to each other.

Thus, the scalar product (3.10) makes the $sl(2,\mathbb{R})$ algebra isometric to $3d$ Minkowski space.

Similarly to the Minkowski space, a non-zero element A is called time-like if $\langle\, A\, A\,\rangle > 0$, space-like if $\langle\, A\, A\,\rangle < 0$ and light-like if $\langle\, A\, A\,\rangle = 0$. For example, T_0 is time-like, whereas T_1 and T_2 are space like. An example of a light-like element is

$$T_+ = \frac{1}{2}\,(T_0 + T_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ , \tag{3.14}$$

which is a nilpotent matrix: $T_+^2 = 0$.

In general, a bilinear symmetric form on a Lie algebra $\mathcal{G}$ is introduced by the Killing form

$$\mathcal{K}(A,B) = \mathrm{Tr}(\mathrm{ad}_A\,\mathrm{ad}_B)\ , \tag{3.15}$$

where ad_A is the operator for the adjoint representation (3.5). To calculate the Killing form, one can choose a basis in $\mathcal{G}$ and associate to ad_A a matrix $(\mathrm{ad}_A)^m{}_n$ in a standard way (see (E.4)). Then one finds

$$\mathcal{K}(A,B) = (\mathrm{ad}_A)^m{}_n(\mathrm{ad}_B)^n{}_m\ . \tag{3.16}$$

Note that this calculation does not depend on the choice of a basis e_n.

For $A = A^m\, e_m$ and $B = B^n\, e_n$, the Killing form can be written as

$$\mathcal{K}(A,B) = \mathcal{K}_{mn}\, A^m\, B^n\ , \tag{3.17}$$

with

$$\mathcal{K}_{mn} = c^k{}_{ml}\, c^l{}_{nk}\ , \tag{3.18}$$

where $c^k{}_{ml}$ are the structure constants of $\mathcal{G}$ (see the exercises 1 and 2).

This calculation for the $sl(2,\mathbb{R})$ algebra gives

$$\mathcal{K}(A,B) = -8\langle\, A\, B\,\rangle\ . \tag{3.19}$$

Exercises 7-9.

3. The exponential map

Due to (3.8), the square of any $A \in sl(2,\mathbb{R})$ is proportional to the unit matrix

$$A \cdot A = -\langle\, A\, A\,\rangle\, I\ . \tag{3.20}$$

This formula helps to find a compact form of e^A

$$e^A = \cos\theta\ I + \sin\theta\ \hat{A}\ , \qquad \text{with} \quad \theta = \sqrt{\langle\, A\, A\,\rangle}\ , \quad \hat{A} = \frac{A}{\theta}\ , \quad \text{if} \quad \langle\, A\, A\,\rangle > 0\ ; \tag{3.21}$$

$$e^A = \cosh\lambda\ I + \sinh\lambda\ \hat{A}\ , \quad \text{with} \quad \lambda = \sqrt{-\langle\, A\, A\,\rangle}\ , \quad \hat{A} = \frac{A}{\lambda}\ , \quad \text{if} \quad \langle\, A\, A\,\rangle < 0\ ; \tag{3.22}$$

$$e^A = I + A\ , \qquad \text{if} \quad \langle\, A\, A\,\rangle = 0\ . \tag{3.23}$$

In particular,

$$e^{\theta\, T_0} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\ , \qquad e^{\lambda\, T_2} = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}\ , \qquad e^{\rho\, T_+} = \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix}\ . \tag{3.24}$$

Exercise 10.

4. The adjoint representation of $SL(2,\mathbb{R})$

The $SL(2,\mathbb{R})$ group is the set of 2×2 real matrixes with unit determinant and the standard multiplication rule of matrices. An element $g \in SL(2,\mathbb{R})$ and its inverse g^{-1} can be written as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ , \qquad g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\ , \qquad \text{with} \quad ad - bc = 1\ . \tag{3.25}$$

Since $\det e^A = e^{\mathrm{Tr}\, A}$, the exponentials (3.21)-(3.23) have the unit determinant, but note that this map covers only a part of $SL(2,\mathbb{R})$. In particular, one can not get the elements with $\mathrm{Tr}\, g < -2$.

Let us introduce the operators Ad_g, which act on $sl(2,\mathbb{R})$ by

$$\mathrm{Ad}_g(A) = g\, A\, g^{-1}\ . \tag{3.26}$$

It is a representation of $SL(2,\mathbb{R})$ ($\mathrm{Ad}_g\, \mathrm{Ad}_{g'} = \mathrm{Ad}_{g\, g'}$), which is called the adjoint representation.

The transformations (3.26) obviously leave the bilinear form (3.10) invariant

$$\langle\, \mathrm{Ad}_g(A)\ \mathrm{Ad}_g(B)\rangle = \langle\, A\ B\rangle\ . \tag{3.27}$$

The corresponding transformations of the coordinates $A^n = \langle\, T^n\, A\,\rangle$ are

$$A^n \mapsto \Lambda^n{}_m\, A^m\ , \tag{3.28}$$

with

$$\Lambda^n{}_m = \langle\, T^n\, g\, T_m\, g^{-1}\,\rangle\ . \tag{3.29}$$

Since the $sl(2,\mathbb{R})$ algebra is isometric to $3d$ Minkowski space, we find that (3.28) is a $3d$ Lorentz transformation. Thus, eq. (3.29) provides a map from the $SL(2,\mathbb{R})$ group to the Lorentz group. As far as the $SL(2,\mathbb{R})$ group manifold is connected (see the next section), the matrices $\Lambda^n{}_m$ belong to the $SO_{\uparrow}(1,2)$ subgroup. Note that (3.29) maps g and $-g$ to the same Lorentz matrix.

Exercises 11-12.

5. Coordinates on $SL(2,\mathbb{R})$

$SL(2,\mathbb{R})$ is a three dimensional manifold. Its parameterization can be obtained as a combination of one parameter subgroups of the type (3.24). One of these combinations (see the exercise 10) can be written as

$$g(\lambda,\alpha,\rho) = \begin{pmatrix} \cosh\lambda\,\cos\alpha + \sinh\lambda\,\cos\beta & -\cosh\lambda\,\sin\alpha + \sinh\lambda\,\sin\beta \\ \cosh\lambda\,\sin\alpha + \sinh\lambda\,\sin\beta & \cosh\lambda\,\cos\alpha - \sinh\lambda\,\cos\beta \end{pmatrix} . \tag{3.30}$$

These parameters become 'global coordinates' on $SL(2,\mathbb{R})$ for

$$\lambda \geq 0, \qquad 0 \leq \alpha < 2\pi \;, \qquad 0 \leq \beta < 2\pi \;. \tag{3.31}$$

To prove this statement, we consider another useful parameterization.

Adding the unit matrix to the basis (3.7), one gets a basis in the space of 2×2 matrices. Therefore, any $g \in SL(2,\mathbb{R})$ can be written as

$$g = c\,I + u^n\,T_n \;, \tag{3.32}$$

with some real numbers c and u^n. The matrix form of (3.32) is

$$g = \begin{pmatrix} c - u_2 & -u_0 - u_1 \\ u_0 - u_1 & c + u_2 \end{pmatrix} , \tag{3.33}$$

where the parameters $(u_n,\, c)$ satisfy the condition

$$c^2 + (u_0)^2 - (u_1)^2 - (u_2)^2 = 1 \;. \tag{3.34}$$

To parameterize g in terms of independent variables it is convenient to consider the planes $(u_1,\, u_2)$ and $(u_0,\, c)$. A point from the plane $(u_1,\, u_2)$ fixes the radial distance on the plane $(u_0,\, c)$ by $c^2 + (u_0)^2 = 1 + (u_1)^2 + (u_2)^2$ and the freedom remains in the polar angle only. Thus, the $SL(2,\mathbb{R})$ group manifold can be treated as $\mathbb{R}^2 \times S^1$, where $\mathbb{R}^2$ stands for the plane $(u_1,\, u_2)$ and S^1 for the polar angle on $(u_0,\, c)$.

Comparing (3.33) and (3.30) we find

$$u_1 = -\sinh\lambda\,\sin\beta \;, \qquad u_0 = \cosh\lambda\,\sin\alpha \;, \tag{3.35}$$

$$u_2 = -\sinh\lambda\,\cos\beta \;, \qquad c = \cosh\lambda\,\cos\alpha \;. \tag{3.36}$$

Hence, $\sinh\lambda$ and β play the role of polar coordinates on the plane $(u_1,\, u_2)$ and α is the polar angle on $(u_0,\, c)$.

Exercises 13-14.

6. Functions, vector fields, 1-forms and the metric on $SL(2,\mathbb{R})$

The matrix elements $g_{\alpha\beta}$ are functions on the $SL(2,R)$ group manifold, which are related by $g_{11}\,g_{22} - g_{12}\,g_{21} = 1$.

Vector fields $\hat{V}$ are given as linear operators acting on functions (see the appendix)

$$\hat{V}[f] = V^\mu(x)\frac{\partial f(x)}{\partial x^\mu}\ . \tag{3.37}$$

Here x^μ are coordinates and $V^\mu(x)$ denote the components of $\hat{V}$ in these coordinates. The solutions of the equation $\dot{x}^\mu = V^\mu(x)$ define a flow on the manifold.

Let us introduce two vector fields $\hat{L}_A$ and $\hat{R}_A$, labeled by $A \in sl(2,\mathbb{R})$ and defined as

$$\hat{L}_A(g_{\alpha\beta}) = (A\,g)_{\alpha\beta}\ , \qquad \hat{R}_A(g_{\alpha\beta}) = (g\,A)_{\alpha\beta}\ . \tag{3.38}$$

Writing this equations without matrix indices

$$\hat{L}_A(g) = A\,g\ , \qquad \hat{R}_A(g) = g\,A\ , \tag{3.39}$$

one immediately finds the corresponding flows (see the appendix for a definition)

$$g \mapsto g(t) = e^{tA}\,g\ , \qquad g \mapsto g(t) = g\,e^{tA}\ , \tag{3.40}$$

as left and right multiplications for $\hat{L}_A$ and $\hat{R}_A$, respectively.

As it follows from (3.39)

$$[\hat{L}_A, \hat{L}_B] = -\hat{L}_{[A,B]}\ , \qquad [\hat{R}_A, \hat{R}_B] = \hat{R}_{[A,B]}\ , \qquad [\hat{L}_A, \hat{R}_B] = 0\ . \tag{3.41}$$

If A and B are the basis vectors (3.7), these commutators become

$$[\hat{L}_m, \hat{L}_n] = -2\epsilon^l{}_{mn}\,\hat{L}_l\ , \qquad [\hat{R}_m, \hat{R}_n] = 2\epsilon^l{}_{mn}\,\hat{R}_l\ , \qquad [\hat{L}_m, \hat{R}_n] = 0\ , \tag{3.42}$$

where we have used the notations $\hat{L}_n = \hat{L}_{T_n}$, $\hat{R}_n = \hat{R}_{T_n}$.

1-forms $\theta = \theta_\mu(x)\,dx^\mu$ are characterized by co-vector fields $\theta_\mu(x)$. They act on vector fields and give functions

$$\theta\,[\,\hat{V}\,] = \theta_\mu(x)\ V^\mu(x)\ . \tag{3.43}$$

Particular examples of 1-forms are differentials of functions df and the rule (3.43) provides

$$df[\hat{V}] = \hat{V}[f]\ . \tag{3.44}$$

Similarly to the vector fields, let us introduce the left and the right 1-forms parameterized by the elements of $sl(2,\mathbb{R})$

$$L_A = \langle A\,dg\,g^{-1}\rangle\ , \qquad R_A = \langle A\,g^{-1}\,dg\rangle\ . \tag{3.45}$$

The action of these 1-forms on the vector fields (3.39) are given by

$$L_A[\hat{L}_B] = \langle A\,B\rangle\ , \qquad R_A[\hat{R}_B] = \langle A\,B\rangle\ , \tag{3.46}$$

and

$$L_A[\hat{R}_B] = \langle A\,g\,B\,g^{-1}\rangle\ , \qquad R_A[\hat{L}_B] = \langle A\,g^{-1}\,B\,g\,\rangle\ . \tag{3.47}$$

Taking A and B as the basis vectors (3.7) and using the notations $L_m = L_{T_m}$, $R_m = R_{T_m}$ we get

$$L_m[\hat{L}_n] = \eta_{mn} \ , \qquad R_m[\hat{R}_n] = \eta_{mn} \ . \tag{3.48}$$

Thus, the left and right 1-forms are dual to the corresponding vector fields. One also has

$$L_m[\hat{R}_n] = \langle T_m \, g \, T_n \, g^{-1} \rangle \ , \qquad R_m[\hat{L}_B] = \langle T_m \, g^{-1} \, T_n \, g \, \rangle \ . \tag{3.49}$$

The last two functions are just the matrix components of Lorentz transformations (3.29).

One can also consider the left and the right 1-forms with values in the $sl(2, \mathbb{R})$ algebra (Maurer-Cartan form)

$$L_n \, T^n = dg \, g^{-1} \ , \qquad R_n T^n = \ g^{-1} dg \ . \tag{3.50}$$

Here we have used the summation property (E.12) for the matrices T_n.

The metric tensor on $SL(2.\mathbb{R})$ is defined by

$$\mathrm{g} = \langle \, (g^{-1} \, dg) \, \otimes \, (g^{-1} \, dg) \rangle \ . \tag{3.51}$$

It is a symmetric 2-form with the components

$$\mathrm{g}_{\mu\nu} = \langle \, g^{-1} \, \partial_\mu \, g \, g^{-1} \, \partial_\nu \, g \rangle \ . \tag{3.52}$$

In terms of the left or the right 1-forms the metric (3.51) reads

$$\mathrm{g} = L^n \otimes L_n = R^n \otimes R_n \ , \tag{3.53}$$

and the components are

$$\mathrm{g}_{\mu\nu} = \eta^{mn} \, L_{m, \ \mu} \, L_{n}, \ \nu = \eta^{mn} \, R_{m, \ \mu} \, R_{n}, \ \nu \ , \tag{3.54}$$

with

$$L_{m, \ \mu} = \langle T_m \, \partial_\mu g \, g^{-1} \rangle \ , \qquad R_{m, \ \mu} = \langle T_m \, g^{-1} \, \partial_\mu g \rangle \ . \tag{3.55}$$

The metric tensor (3.52) is invariant under the left ($g \mapsto hg$) and the right ($g \mapsto gh$) multiplications of g by the group elements $h \in SL(2, \mathbb{R})$.

Exercises 15.

Remarks

A Lie algebra $\mathcal{G}$ is called semi-simple if its Killing form is non-degenerated.

A Lie algebra $\mathcal{G}$ is called Abelian, if $[A, B] = 0$ for any $A, \ B \in \mathcal{G}$.

Examples of semi-simple Lie algebras are $o(n)$, for $n > 2$, $su(N)$, $sp(n)$, $o(p, q)$, $su(p, q)$.

Exercise 16.

Appendix

Covariant tensors $T(x)$ have down components $T_{mn...}(x)$. They act on vector fields

$$T(v, w, ...) = T_{mn...}\, v^m\, w^n ... \ . \tag{A.1}$$

Contravariant tensors $T(x)$ have upper components $T^{mn...}(x)$. They act on functions

$$T(f, g, ...) = T^{mn...}\, \partial_m f\, \partial_n g ... \ . \tag{A.2}$$

Functions, forms and metric are covariant tensors, while vector fields are contravariant ones.
A map $\phi: \mathcal{M} \mapsto \tilde{\mathcal{M}}$ from a manifold $\mathcal{M}$ to a manifold $\tilde{\mathcal{M}}$ creates:

- ϕ^* - pull-back map of covariant tensors.
- ϕ_* - push-forward map of contravariant tensors.

One has

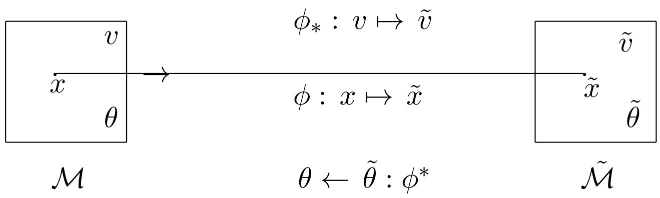
$$\phi(x) = \tilde{x} \ , \qquad\qquad \phi^*\tilde{f} = \tilde{f} \circ \phi \ , \tag{A.3}$$

$$\phi_* v\, (\tilde{f}) = v(\phi^*\tilde{f}) \ , \qquad \phi^*\tilde{\theta}(v) = \tilde{\theta}(\phi_* v) \ . \tag{A.4}$$

Here $\tilde{x}$ is a point in $\tilde{\mathcal{M}}$, $\tilde{f}$ is a function on $\tilde{\mathcal{M}}$, $\tilde{f} \circ \phi$ denotes the composition of $\tilde{f}$ and ϕ; v is a vector field on $\mathcal{M}$ and $\tilde{\theta}$ is a co-vector field on $\tilde{\mathcal{M}}$. The generalization to higher order tensors is straightforward. Then, one gets in components

$$(\phi^*\tilde{T})_{mn} = \frac{\partial \tilde{x}^{\tilde{m}}}{\partial x^m} \frac{\partial \tilde{x}^{\tilde{n}}}{\partial x^n}\, \tilde{T}_{\tilde{m}\tilde{n}} \ , \tag{A.5}$$

$$(\phi_* T)^{\tilde{m}\tilde{n}} = \frac{\partial \tilde{x}^{\tilde{m}}}{\partial x^m} \frac{\partial \tilde{x}^{\tilde{n}}}{\partial x^n}\, T^{mn} \ . \tag{A.6}$$



The pull-back map of the metric tensor is called the induced metric.

A flow ϕ_t on $\mathcal{M}$ is a one parameter additive family of transformations of $\mathcal{M}$

$$\phi_t : \mathcal{M} \mapsto \mathcal{M} \ , \qquad \text{with} \qquad \phi_t \circ \phi_{t'} = \phi_{t+t'} \ . \tag{A.7}$$

A vector field $v(x)$ provides the dynamical equation $\dot{x} = v(x)$. The solutions of this equation $x(t)$ with all possible initial data $x(t)|_{t=0} = x$ define a flow $\phi^v_t(x) = x(t)$. This flow acts on covariant and contravariant tensors according to (A.5) and (A.6), respectively. The Lie derivative $\mathcal{L}_v$ of a tensor filed T is defined by

$$\mathcal{L}_v\, T = \frac{d}{dt}\, (\phi^{v*}_t T)\, |_{t=0} \ . \tag{A.8}$$

The components of $\mathcal{L}_v T$ (see the next lecture) can be obtained from (A.5), replacing ϕ^* by ϕ^{v*}_t, and using that

$$\phi^v_t(x) = x + tv(x) + O(t^2),$$

for small t.

Exercises

1. Let e_n be a basis of a Lie algebra $\mathcal{G}$. Since $[\, e_m\, ,\, e_n\,] \in \mathcal{G}$, one has

$$[\, e_m\, ,\, e_n\,] = e_l\, c^l{}_{mn}\ , \tag{E.1}$$

with some constants $c^l{}_{mn}$, which are called the structure constants of $\mathcal{G}$.

Check that the structure constants satisfy the relations

$$c^l{}_{mn} = -c^l{}_{nm}\ , \tag{E.2}$$
$$c^k{}_{lm}\, c^j{}_{kn} + c^k{}_{mn}\, c^j{}_{kl} + c^k{}_{nl}\, c^j{}_{km} = 0\ . \tag{E.3}$$

2. Let $(\mathrm{ad}_A)^m{}_n$ be the matrix associate with ad_A in a basis e_n

$$\mathrm{ad}_A(e_n) = e_m\, (\mathrm{ad}_A)^m{}_n\ . \tag{E.4}$$

Check that

$$(\mathrm{ad}_{e_l})^m{}_n = c^m{}_{ln}\ , \tag{E.5}$$

and relate the identity (E.3) to the commutator of the basis vectors in the adjoint representation.

3. Check that the commutators of the basis vectors (3.7) are

$$[\, T_1,\, T_2\,] = 2\, T_0\ , \qquad [\, T_1,\, T_0\,] = 2\, T_2\ , \qquad [\, T_0,\, T_2\,] = 2\, T_1\ . \tag{E.6}$$

4. Verify the following relation

$$(T_n)_{\alpha\beta}\ (T^n)_{\alpha'\beta'}\ =\ \delta_{\alpha\beta}\, \delta_{\alpha'\beta'} - 2\delta_{\alpha\beta'}\, \delta_{\beta\alpha'}\ . \tag{E.7}$$

5. Prove that

$$\epsilon_{ijk}\ \epsilon_{lmn} = \det \begin{pmatrix} \eta_{il} & \eta_{im} & \eta_{in} \\ \eta_{jl} & \eta_{jm} & \eta_{jn} \\ \eta_{kl} & \eta_{km} & \eta_{kn} \end{pmatrix}\ . \tag{E.8}$$

6. Using (E.8) derive the summation rules

$$\epsilon_{ij}{}^k\ \epsilon_{kmn} = \eta_{im}\, \eta_{jn} - \eta_{in}\, \eta_{jm}\ , \qquad \epsilon_i{}^{jk}\ \epsilon_{jkn} = 2\eta_{in}\ , \qquad \epsilon^{ijk}\ \epsilon_{ijk} = 6\ . \tag{E.9}$$

7. By (3.8) check that

$$\langle\, T_l\, T_m\, T_n\, \rangle = \epsilon_{lmn}\ . \tag{E.10}$$

8. Using (E.9) derive that

$$\langle\, T_k\, T_l\, T_m\, T_n\, \rangle = \eta_{km}\, \eta_{ln} - \eta_{kn}\, \eta_{lm} - \eta_{kl}\, \eta_{mn}\ . \tag{E.11}$$

9. Let $A \in sl(2,\mathbb{R})$ and $B \in sl(2,\mathbb{R})$. Using (E.7) check that

$$\langle\, T_n\, A\, \rangle\, \langle\, T^n\, B\, \rangle = \langle\, A\, B\, \rangle\ . \tag{E.12}$$

10. Calculate the product of three exponents given below and show that

$$e^{\theta\, T_0}\, e^{\lambda\, T_2}\, e^{\gamma\, T_0} = \begin{pmatrix} \cosh\lambda\, \cos\alpha + \sinh\lambda\, \cos\beta & -\cosh\lambda\, \sin\alpha + \sinh\lambda\, \sin\beta \\ \cosh\lambda\, \sin\alpha + \sinh\lambda\, \sin\beta & \cosh\lambda\, \cos\alpha - \sinh\lambda\, \cos\beta \end{pmatrix}\ , \tag{E.13}$$

with $\alpha = \theta + \gamma$ and $\beta = \theta - \gamma$.

11. Prove that:
a) any $g \in SL(2,\mathbb{R})$ with $-2 < \operatorname{Tr} g < 2$ is given by (3.21).
b) any $g \in SL(2,\mathbb{R})$ with $\operatorname{Tr} g > 2$ is given by (3.22).
c) any $g \in SL(2,\mathbb{R})$ with $\operatorname{Tr} g < -2$ is given by $g = -e^A$ with a space-like A.
d) any $g \in SL(2,\mathbb{R})$ with $\operatorname{Tr} g = 2$ is given by (3.23).

12. Using (E.12), prove that the matrixes $\Lambda^n{}_m$ defined by (3.29) satisfy the conditions

$$\Lambda^n{}_m \, \Lambda_{n\,l} = \eta_{ml} \ . \tag{E.14}$$

13. Prove that the inverse to (3.32) is

$$g^{-1} = c\, I - u^n\, T_n \ . \tag{E.15}$$

14. The $SU(1,1)$ group is defined as the set of 2×2 complex matrixes with unit determinant which preserve the following scalar product

$$(\,\psi|\chi\,) = \psi_1^*\,\chi_1 - \psi_2^*\,\chi_2 \ . \tag{E.16}$$

Show that:
a) a group element $\tilde{g} \in SU(1,1)$ is given by

$$\tilde{g} = \begin{pmatrix} z & u \\ u^* & z^* \end{pmatrix} \ , \tag{E.17}$$

where the z and u are complex numbers, with $|z|^2 - |u|^2 = 1$.
b) the map

$$\tilde{g} \mapsto \ g = U\,\tilde{g}\,U^{-1} \qquad \text{with} \qquad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \ , \tag{E.18}$$

transforms the $SU(1,1)$ matrixes to the $SL(2,\mathbb{R})$ ones. Thus, these two groups are equivalent.

15. Let x^n be the normal coordinates on $SL(2,\mathbb{R})$ defined by

$$g = e^{x^n\, T_n} \ . \tag{E.19}$$

Calculate in theses coordinates :
a) The components of the vector fields $\hat{L}_n$ and $\hat{R}_n$.
b) The components of the co-vector fields for the 1-forms L_n and R_n.
c) The components of the metric tensor g.

16. Check that the Killing form is non-degenerated for the $su(2)$ algebra, but it is degenerated for $u(2)$.

17. Let us consider two maps from the $sl(2,\mathbb{R})$ algebra to the $SL(2,\mathbb{R})$ group

$$A \mapsto g = e^A \qquad \text{and} \qquad B \mapsto g = -e^B \ . \tag{E.20}$$

Check that they together cover $SL(2,\mathbb{R})$. Describe the domain on $SL(2\mathbb{R})$, where these maps intersect and find there the relation between A and B.

18. Check that the Lie derivative of the metric tensor on the $SL(2,\mathbb{R})$ group manifold vanishes for the left and right vector fields.

19. Let us consider the 4-dimensional space $\mathbb{R}^{2,2}$ with coordinates $(u_1,\, u_2,\, u_0,\, c)$ and the metric

$$ds^2 = dc^2 + du_0^2 - du_1^2 - du_2^2 \ . \tag{E.21}$$

Its isometry group is $O(2,2)$. The hyperboloid

$$c^2 + u_0^2 - u_1^2 - u_2^2 = 1 \tag{E.22}$$

embedded in $\mathbb{R}^{2,2}$ is invariant under the $O(2,2)$ transformations. Verify that the induced metric on the hyperboloid (E.22) coincides with the metric on $SL(2,\mathbb{R})$.

References

[1] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov. Modern Geometry-Methods and Applications Vol. I. Springer-Verlag New York. 1984
(Chapter 3, PP 24, Pages 212-224)

Lecture 4

Particle dynamics on manifolds

In this lecture we consider simple models of particle dynamics and demonstrate how the mathematical tools introduced in the previous lectures work for them.

1. The Liouville model

The first example is the dynamics of a particle in the exponential potential. Here we obtain the space of motions M, calculate the symplectic form on M and find its tangent vectors.

The Liouville field theory is described by the dynamical equation

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) \varphi(\tau, \sigma) + 4m^2\, e^{2\varphi(\tau,\sigma)} = 0 \ , \tag{4.1}$$

where $m > 0$ is a coupling constant and $(\sigma,\, \tau)$ are space-time coordinates. This model of exponentially self-interacting field theory is integrable. Its general solution in terms of two arbitrary functions A and B can be written in the form

$$\varphi(\tau, \sigma) = \frac{1}{2}\, \log \frac{A'(\tau+\sigma)\, B'(\tau-\sigma)}{[1 + m^2\, A(\tau+\sigma)\, B(\tau-\sigma)]^2} \ . \tag{4.2}$$

Let us consider the homogeneous field configurations $\partial_\sigma \varphi = 0$. These fields are time dependent $\varphi(\tau, \sigma) \equiv x(\tau)$ and they describe the dynamics of a particle in the exponential potential

$$\ddot{x}(\tau) + 4m^2\, e^{2x(\tau)} = 0 \ . \tag{4.3}$$

The Lagrangian of this model is

$$L = \frac{1}{2}\, \dot{x}^2 - 2m^2\, e^{2x} \ , \tag{4.4}$$

and it provides the standard symplectic form on TQ (see (2.57))

$$\omega_L = dv \wedge dx \ . \tag{4.5}$$

Using the conservation of energy

$$E = \frac{1}{2}\, \dot{x}^2 + 2m^2\, e^{2x} \ , \tag{4.6}$$

the equation of motion (4.3) can be integrated in the form

$$x(t) = \log \frac{p}{2m\, \cosh(q + p\tau)} \ , \tag{4.7}$$

where $p = \sqrt{2E}$ and q is an integration constant. The variables (p, q) parameterize the space of motions and since the energy is positive, these variables are given on the half-plane $p > 0$. Note that the solution (4.7) is obtained from (4.2) for $m\, A(x) = m\, B(x) = e^{(q+px)}$.

The symplectic form on the space of motions (4.7) is induced by (4.5) as the pull-back map (see the appendix of Lecture 3 and (2.63)) for

$$\phi(t_0) : x(t) \mapsto (x, v) = (x(t_0), \dot{x}(t_0)) \ . \tag{4.8}$$

Calculating first the velocity from (4.7)

$$\dot{x}(t) = -p\, \tanh(q + p\tau) \ , \tag{4.9}$$

and then taking the differentials of (4.7) and (4.9) with respect to p and q

$$dx(t) = \frac{dp}{p} - \tanh(q + p\tau)\,(dq + \tau\, dp)\ , \tag{4.10}$$

$$d\dot{x}(t) = -\tanh(q + p\tau)\, dp - \frac{p}{\cosh^2(q + p\tau)}\,(dq + \tau\, dp)\ , \tag{4.11}$$

by direct calculation we obtain

$$d\dot{x}(t_0) \wedge dx(t_0) = dp \wedge dq\ . \tag{4.12}$$

As it was expected, this symplectic form does not depend on t_0.

The equation for the tangent vectors (2.65) in this case reads

$$\ddot{u}(\tau) + \frac{2p^2}{\cosh^2(q + p\tau)}\, u(\tau) = 0\ . \tag{4.13}$$

This second order linear equation has two linear independent solutions with the unit Wronskian

$$u_1(\tau) = \tanh(q + p\tau) \qquad \text{and} \qquad u_2(\tau) = \tau\, \tanh(q + p\tau) - 1/p\ . \tag{4.14}$$

This tangent vectors are also obtained from (4.7), differentiating it with respect to q and p.

Exercise 1-2.

2. A free particle in a curved space

In this section we consider the dynamics of a free particle in a curved space and show how the Hamilton equations on TQ reproduce the equations for geodesics.

Let us consider N-dimensional configuration space Q with coordinates q^μ ($\mu = 1, ..., N$) and the matric tensor $\mathrm{g}_{\mu\nu}(q)$. A free particle dynamics in this space is described by the Lagrangian

$$L = \frac{1}{2}\, \mathrm{g}_{\mu\nu}(q)\, \dot{q}^\mu\, \dot{q}^\nu\ . \tag{4.15}$$

The metric tensor is assumed positively defined to realize the minimal action principle. Multiplying the Euler-Lagrange equations obtained from (4.15) by $\mathrm{g}^{\mu\nu}(q)$, which is the inverse to the metric tensor ($\mathrm{g}^{\mu\mu'}\mathrm{g}_{\mu'\nu} = \delta^\mu{}_\nu$), one finds the equations for geodesics

$$\ddot{q}^\mu + \Gamma^\mu_{\nu\sigma}(q)\, \dot{q}^\nu\, \dot{q}^\sigma = 0\ . \tag{4.16}$$

Here $\Gamma^\mu_{\nu\sigma}(q)$ are the Christoffel symbols

$$\Gamma^\mu_{\nu\sigma}(q) = \frac{1}{2}\, \mathrm{g}^{\mu\mu'} \left(\partial_\nu \mathrm{g}_{\mu'\sigma} + \partial_\sigma \mathrm{g}_{\mu'\nu} - \partial_{\mu'} \mathrm{g}_{\nu\sigma}\right)\ . \tag{4.17}$$

The Lagrangian (4.15) provides on TQ the following symplectic form (see (2.57))

$$\omega_L = \mathrm{g}_{\mu\nu}\, dv^\nu \wedge dq^\mu + \partial_\sigma \mathrm{g}_{\mu\nu}\, v^\nu\, dq^\sigma \wedge dq^\mu\ , \tag{4.18}$$

and the Hamiltonian (see (2.54))

$$H = \frac{1}{2}\, \mathrm{g}_{\mu\nu}(q)\, v^\mu\, v^\nu\ . \tag{4.19}$$

Differentiating this Hamiltonian, one can solve the equation $dH + V_H\rfloor\omega_L = 0$ for the Hamiltonian vector field V_H and obtain

$$V_H = v^\mu \frac{\partial}{\partial q^\mu} + \frac{1}{2}\,\mathrm{g}^{\mu\mu'}\left(\partial_{\mu'}\mathrm{g}_{\nu\sigma} - 2\partial_\sigma \mathrm{g}_{\mu'\nu}\right) v^\nu\, v^\sigma\, \frac{\partial}{\partial v^\mu}\ . \tag{4.20}$$

The corresponding Hamilton equations

$$\dot q^a = v^a\ , \qquad\qquad \dot v^a = -\Gamma^a_{bc}(q)\, v^b\, v^c\ , \tag{4.21}$$

are indeed equivalent to the equations of geodesics (4.16), written in the first order form.

Exercise 3-5.

3. Particle dynamics in $SU(2)$

Now we consider the dynamics of a particle on the $SU(2)$ group manifold. We integrate the equations of motion and describe the symplectic and Poisson bracket structures of the system. Some details on the $SU(2)$ group manifold are given in the Appendix (see also the exercise 6).

The $SU(2)$ group manifold geometrically is realized as the 3-dimensional sphere S^3. The metric tensor on $SU(2)$ is defined similarly to the $SL(2,R)$ case by (A.10). Then, the free particle Lagrangian (4.15) on $SU(2)$ takes the form

$$L = \frac{1}{2}\,\langle g^{-1}\dot g g^{-1}\dot g\rangle\ . \tag{4.22}$$

The Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot g_{\alpha\beta}}\right) = \left(\frac{\partial L}{\partial g_{\alpha\beta}}\right)\ , \tag{4.23}$$

can be reduced to the matrix equation

$$\frac{d}{dt}\left(g^{-1}\dot g\right) = 0\ . \tag{4.24}$$

Here we have used that

$$\frac{\partial g^{-1}_{\alpha'\beta'}}{\partial g_{\alpha\beta}} = -g^{-1}_{\alpha'\alpha}\, g^{-1}_{\beta\beta'}\ , \tag{4.25}$$

which is a consequence of $\left(g^{-1}g\right)_{\alpha\beta} = \delta_{\alpha\beta}$, and also

$$\frac{\partial L}{\partial \dot g_{\alpha\beta}} = -\frac{1}{2}\left(g^{-1}\dot g g^{-1}\right)_{\beta\alpha}\ , \qquad \frac{\partial L}{\partial g_{\alpha\beta}} = \frac{1}{2}\left(g^{-1}\dot g g^{-1}\dot g g^{-1}\right)_{\beta\alpha}\ . \tag{4.26}$$

Integrating (4.24), first one finds

$$g^{-1}\dot g = r\ , \tag{4.27}$$

with a time independent $su(2)$ matrix r. Further integration of $\dot g = g\, r$ leads to

$$g(t) = g_{_0}\, e^{rt}\ , \tag{4.28}$$

where $g_{_0}$ is the initial value for $g(t)$: $g_{_0} = g(0)$.

Eq. (4.28) defines the space of motions M and it is parameterized by the pair $(g_{_0}, r)$.

To find the symplectic form of the system one can introduce an equivalent to (4.22) Lagrangian written in the first order form

$$\tilde{L} = \langle rg^{-1}\,\dot{g}\rangle - \frac{1}{2}\langle r\,r\rangle\ . \tag{4.29}$$

The variation of (4.29) with respect to r gives $r = g^{-1}\dot{g}$ and plug in it back into (4.29), one indeed gets (4.22). The first order formalism leads to the Hamiltonian formulation. By (4.29) one can introduce the 1-form

$$\theta = \langle rg^{-1}\,dg\rangle\ , \tag{4.30}$$

called the symplectic potential. Its differential is the symplectic form

$$\omega = \langle dr \wedge g^{-1}\,dg\rangle - \langle r\,g^{-1}\,dg \wedge g^{-1}\,dg\rangle\ , \tag{4.31}$$

which can be rewritten as

$$\omega = dr_m \wedge R_m - \epsilon_{klm}\,r_k\,R_l \wedge R_m\ . \tag{4.32}$$

Here

$$r_m = \langle e_m\,r\rangle\ , \qquad R_m = \langle e_m\,g^{-1}\,dg\rangle\ , \tag{4.33}$$

e_m $(m = 1, 2, 3)$ are the basis vectors of the $su(2)$ algebra and, in addition, we have used eq. (A.7) for the normalized traces.

The Hamiltonian vector field V_{r_n}, associated with r_n, is obtained from the equation

$$dr_n + V_{r_n}\rfloor\omega = 0\ , \tag{4.34}$$

which by (4.32) leads to

$$dr_n + V_{r_n}(r_m)\,R_m - R_m(V_{r_n})\,dr_m - \epsilon_{klm}\,r_k\,R_l(V_{r_n})\,R_m + \epsilon_{klm}\,r_k\,R_m(V_{r_n})\,R_l = 0\ . \tag{4.35}$$

Since dr_m and R_m are six linearly independent 1-forms, we obtain

$$R_m(V_{r_n}) = \delta_{mn}\ , \qquad V_{r_n}(r_m) = 2\epsilon_{knm}\,r_k\ . \tag{4.36}$$

The second equation here provides the Poisson brackets (see (2.45))

$$\{r_n, r_m\} = 2\epsilon_{knm}\,r_k\ , \tag{4.37}$$

and the first one leads to (see (4.33)) $V_{r_n}(g_{\alpha\beta}) = (g\,e_n)_{\alpha\beta}$, or equivalently

$$\{r_n, g\} = g\,e_n\ . \tag{4.38}$$

The equation for the Hamiltonian vector field $V_{g_{\alpha\beta}}$, given by $dg_{\alpha\beta} + V_{\alpha\beta}\rfloor\omega = 0$, yields the relations $R_n(V_{g_{\alpha\beta}}) = 0$, which defines the Poisson brackets

$$\{g_{\alpha\beta},\, g_{\alpha'\beta'}\} = 0\ . \tag{4.39}$$

The generalization of this scheme to any semi simple Lie group is straightforward. Then one concludes that the dynamics of a free particle on a semi-simple Lie group G is described by the phase space $G \times \mathcal{G}$, with the pre-symplectic form (4.30). The Poisson brackets of the corresponding variables (g, r) are given by (4.39), (4.38) and (4.37), replacing there $2\epsilon_{knm}$ by the structure constants of $\mathcal{G}$.

Exercises 6-10.

4. A relativistic particle in AdS spaces

A manifold $\mathcal{M}$ with a Lorentzian metric can be interpreted as a curved space-time. The dynamics of a relativistic particle on $\mathcal{M}$ is described by the action proportional to the length of a timelike trajectory. Typical examples of such manifolds are de Sitter (dS) and anti de Sitter (AdS) spaces, which have a constant curvature. This constant is positive for the dS spaces and negative for AdS. The corresponding metric tensor satisfies the Einstein equation with a cosmological constant.

The $(N+1)$-dimensional de Sitter space is defined as the following hyperboloid of a radius R

$$Y_0^2 - \sum_{n=1}^{N+1} Y_n^2 + R^2 = 0 \tag{4.40}$$

embedded in the $(N+2)$-dimensional Minkowski space $\mathbb{R}^{1,\,N+1}$. One can check that the induced metric tensor on the hyperboloid (4.40) has a Lorentzian signature (see the exercise 11), where the space part is given by the N-dimensional sphere and the time coordinate is unbounded.

In this section we concentrate on the AdS spaces. We describe their symmetries and study the dynamics of a relativistic particle.

The $(N+1)$-dimensional AdS space is represented as the hyperboloid

$$X_{0'}^2 + X_0^2 - \sum_{n=1}^{N} X_n^2 = R^2 \tag{4.41}$$

embedded in the $(N+2)$-dimensional flat space $\mathbb{R}^{2,\,N}$ with coordinates X^A, $A = (0', 0, 1, ..., N)$ and the metric tensor $G_{AB} = \text{diag}(+, +, -, ..., -)$. Like in (4.40), the parameter R is called the radius of the hyperboloid. To find the structure of the induced metric, we parameterize the hyperboloid (4.41) by $N+1$ coordinates x^μ $(\mu = 0, 1, ..., N)$

$$\begin{gathered} X^{0'} = r\,\sin\theta\ , \qquad X^0 = r\,\cos\theta\ , \qquad X^n = x^n \quad (n = 1, ..., N)\ , \\ \text{where} \qquad \theta = x^0 \qquad \text{and} \qquad r = \sqrt{R^2 + x^n x^n}\ . \end{gathered} \tag{4.42}$$

The induced metric tensor $g_{\mu\nu} = G_{AB}\partial_\mu X^A \partial_\nu X^B$ has a Lorentzian signature, since

$$g_{00} = r^2\ , \quad g_{0n} = g_{n0} = 0\ , \qquad g_{mn} = -\delta_{mn} + \frac{x_m x_n}{r^2} \tag{4.43}$$

(see also the exercise 12). Due to (4.42)-(4.43), the cyclic coordinate $\theta \in \mathcal{S}^1$ is identified with time. In this way the hyperboloid (4.41) becomes a space-time manifold with a compact time coordinate. Its isometry group is $O(2, N)$ and an analog of the proper Lorentz transformations is the $SO_\uparrow(2, N)$ subgroup, which is represented as a composition of the $SO(2) \times SO(N)$ rotations and the boosts in the planes (X_0, X_n) and $(X_{0'}, X_n)$.

Unwrapping the time coordinate by $\theta \mapsto t \in \mathbb{R}^1$, one gets the space, which is an universal covering of the hyperboloid (4.41). The AdS_{N+1} space is usually associated with this covering space. Since the hyperboloid (4.41) and AdS_{N+1} are locally isometric, particle dynamics on these spaces is similar. The difference is that the closed timelike curves on the hyperboloid (4.41) are not closed in AdS_{N+1}.

Let us consider the 2-dimensional case as an illustrative example. The corresponding hyperboloid $X_{0'}^2 + X_0^2 - X_1^2 = R^2$, embedded in $\mathbb{R}^{2,1}$, is visualized on Fig. 1.

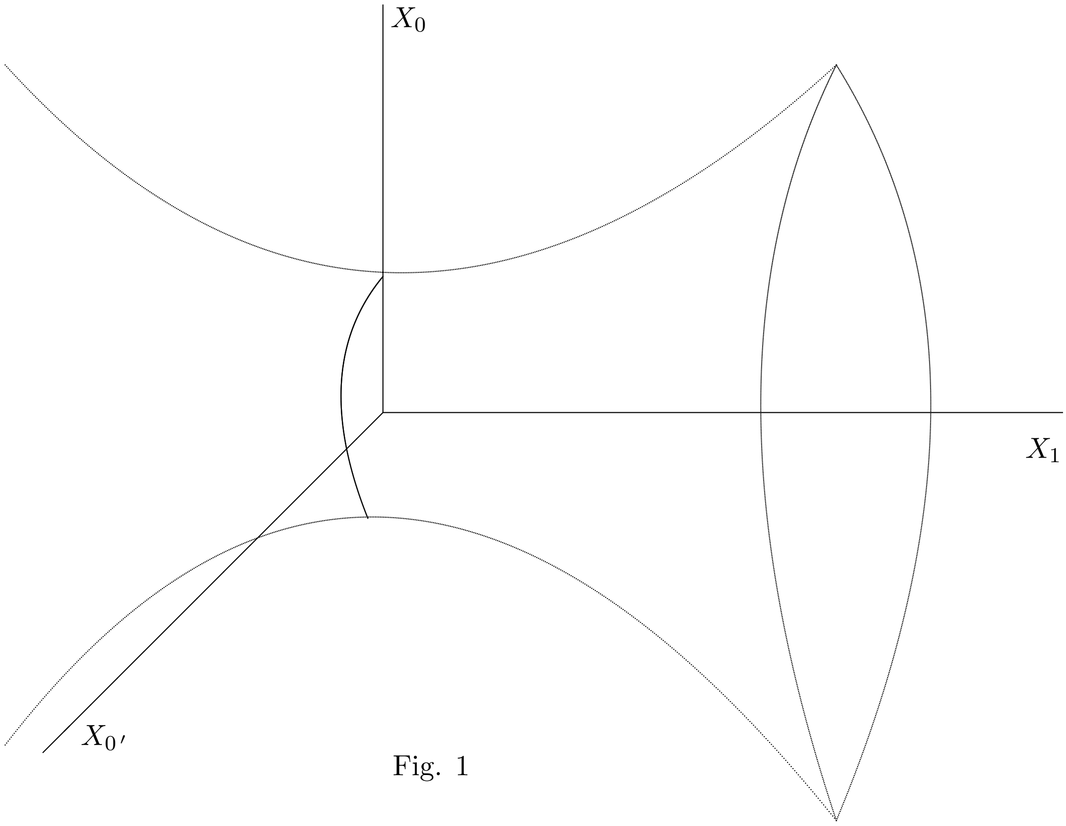


Fig. 1

This hyperboloid can be mapped onto the cylinder (θ, σ), where $\theta \in \mathcal{S}^1$ is the time coordinate (4.42) and $\sigma \in (0, \pi)$ parameterizes X_1 by $X_1 = R \cot \sigma$. Note that the induced metric tensor in the coordinates (θ, σ) becomes conformally flat

$$g_{\mu\nu} = \frac{R^2}{\sin^2 \sigma} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{4.44}$$

Unwrapping the cylinder (θ, σ) one gets AdS_2 as a strip (t, σ), where $t \in \mathbb{R}^1$.

In 3-dimensions the equation for the hyperboloid (4.41)

$$X_{0'}^2 + X_0^2 - X_1^2 - X_2^2 = R^2 \tag{4.45}$$

is equivalent to the condition $ad - bc = 1$ with

$$a = \frac{X_{0'} + X_2}{R} , \quad b = \frac{X_1 + X_0}{R} , \quad c = \frac{X_1 - X_0}{R} , \quad d = \frac{X_{0'} - X_2}{R} . \tag{4.46}$$

Therefore, this space is described by the $SL(2, \mathbb{R})$ group manifold. Moreover, the induced metric on the hyperboloid $dX^A \, dX_A$ coincides with the standard left-right invariant metric on the group manifold $\langle \, g^{-1} dg \, g^{-1} dg \rangle$ (see the exercise 19 of the lecture 3). Note that the left-right symmetry of the metric on $SL(2, \mathbb{R})$ corresponds to the decomposition $SO_{\uparrow}(2, 2) = SO_{\uparrow}(2, 1) \times SO_{\uparrow}(2, 1)$.

The dynamics of a relativistic particle of a mass m moving on the hyperboloid (4.41) can be described by the action

$$S = -\int d\tau \left[\frac{\dot{X}^A \dot{X}_A}{2e} + \frac{em^2}{2} + \frac{\mu}{2}(X^A X_A - R^2) \right] . \tag{4.47}$$

Here e and μ are Lagrange multipliers and τ is an evolution parameter. To have the kinetic term of the space coordinates $\dot{X}_n \dot{X}_n$ with a positive coefficient, one assumes $e > 0$. Note that for $e > 0$

and $m > 0$, after elimination of the Lagrange multipliers e and μ, the action (4.47) reduces to the standard form $S = -ml$, where l is the length of a world-line.

To fix the time direction, we also assume $\dot{\theta} > 0$, which by (4.42) is equivalent to

$$X_0\dot{X}_{0'} - X_{0'}\dot{X}_0 > 0 \ . \qquad (4.48)$$

The $SO_{\uparrow}(2,N)$ symmetry of (4.47) provides the Noether's conserved quantities

$$J_{AB} = P_A\,X_B - P_B\,X_A \ , \qquad (4.49)$$

where P_A are the canonical momenta $P_A = (\partial L)/(\partial \dot{X}^A) = -\dot{X}_A/e$. The dynamical integrals J_{0n} and $J_{0'n}$ are related to the above mentioned boosts, while $J_{00'}$ and J_{mn} to the $SO(2)$ and $SO(N)$ rotations, respectively. We use the notations $J_{0n} = K_n$, $J_{0'n} = L_n$ and $J_{00'} = E$. Since θ is the time coordinate, E is associated with the particle energy, and due to (4.48) it is positive

$$E = P_0X_{0'} - P_{0'}X_0 = e^{-1}(X_0\dot{X}_{0'} - X_{0'}\dot{X}_0) > 0 \ . \qquad (4.50)$$

The dynamical integrals (4.49) allow to represent the set of all trajectories geometrically without solving the dynamical equations. From (4.49) we find N equations as identities in the (P,X)-variables

$$E\,X_n = K_n\,X_{0'} - L_n\,X_0 \ , \qquad (n = 1, ..., N) \ . \qquad (4.51)$$

Since E, K_n, L_n are constants, eq. (4.51) defines a 2-dimensional plane, which goes through the origin of the embedding space $\mathbb{R}^{2,N}$. The intersection of this plane with the hyperboloid (4.41) gives a trajectory, which is a timelike geodesic. This line can be parameterized by the time coordinate θ (see the exercise 13).

The action (4.47) is invariant under reparametrizations $\tau \to f(\tau)$, with the corresponding transformations of the Lagrange multipliers ($\mu \to \mu/f'$, $e \to e/f'$). The gauge symmetry leads to dynamical constraints. Applying the Dirac's procedure, we find three constraints

$$X^AX_A - R^2 = 0 \ , \qquad P_AP^A - m^2 = 0 \ , \qquad P_A\,X^A = 0 \ . \qquad (4.52)$$

They fix the quadratic Casimir number of the symmetry group by

$$\frac{1}{2}\,J_{AB}J^{AB} = m^2R^2 \ . \qquad (4.53)$$

This equation can be written as

$$E^2 + J^2 = K^2 + L^2 + \alpha^2 \ , \qquad (4.54)$$

where

$$J^2 = \frac{1}{2}\,J_{mn}J_{mn} \ , \qquad K^2 = K_nK_n \ , \qquad L^2 = L_nL_n \quad \text{and} \qquad \alpha = mR \ . \qquad (4.55)$$

A set of other quadratic relations follows from (4.49) as the identities in the (P,X)-variables

$$J_{AB}\,J_{A'B'} = J_{AA'}\,J_{BB'} - J_{AB'}\,J_{BA'} \ . \qquad (4.56)$$

These equations are nontrivial in terms of the dynamical integrals, if all indexes A, B, A', B' are different. Taking $A = 0$, $B = 0'$, $A' = m$ and $B' = n$ $(m \neq n)$ we obtain

$$E\,J_{mn} = K_mL_n - K_nL_m \ , \qquad (4.57)$$

and it provides

$$E^2 J^2 = K^2 L^2 - (K \cdot L)^2 \ . \tag{4.58}$$

From eq. (4.54) and (4.58) follows a quadratic equation for E^2 and one finds

$$E^2 = \frac{1}{2}\left(K^2 + L^2 + \alpha^2 + \sqrt{\alpha^4 + 2\alpha^2(K^2 + L^2) + (K^2 - L^2)^2 + 4(KL)^2}\right) \ . \tag{4.59}$$

We neglect the small root of the quadratic equation, since it does not describe a real trajectory. Indeed, using the result of the exercise 13, the small root gives an imaginary $r(\theta)$ in eq. (E.11).

According to (4.59) α is the lowest value of energy. Two other inequalities

$$E^2 \geq K_n K_n \ , \qquad\qquad E^2 \geq L_n L_n \ , \tag{4.60}$$

also follow from (4.59). They are similar to the relation between the energy and momentum in the Minkowski space. On the basis of these inequalities one can show that the choice of the time direction by (4.48) is invariant under the $SO_{\uparrow}(2,N)$ transformations (see the exercise 14).

Eqs. (4.59) and (4.57) define E and J_{mn} as functions of (K_n, L_n) and, therefore, (K_n, L_n) are global coordinates on the space of dynamical integrals J_{AB}. The physical phase space can be identified with the space of dynamical integrals, since they form a complete set of gauge invariant variables. Thus, the 2N variables (K_n, L_n) form global coordinates on the physical phase space.

The canonical brackets $\{P_A, X^B\} = \delta_A{}^B$ provide the $o(2,N)$ algebra of the generators (4.49)

$$\{J_{AB},\, J_{A'B'}\} = G_{AA'} J_{BB'} + G_{BB'} J_{AA'} - G_{AB'} J_{BA'} - G_{BA'} J_{AB'} \ . \tag{4.61}$$

Due to the gauge invariance of J_{AB}, the Poisson bracket relations (4.61) remain the same after reduction to the physical phase space, where the generators E and J_{mn} become non-linear functions of the independent variables (K_n, L_n).

The Poisson brackets on the physical phase space can be obtained by inversion of the symplectic form, which corresponds to the reduced canonical form $dP_A \wedge dX^A$. To calculate the reduction of this canonical form, it is useful to note that on the constraint surface (4.52) the following relation holds

$$dP_A \wedge dX^A = \frac{1}{2\alpha^2} J_{AB}\, dJ^{AC} \wedge dJ^B{}_C \ . \tag{4.62}$$

The righthand side of this equation can be easily recalculated in terms of the coordinates (K_n, L_n). We demonstrate this procedure in 2-dimensions.

In this case there are only three dynamical integrals E, $K = K_1$ and $L = L_1$. They are related by the Casimir condition (4.54)

$$E^2 = K^2 + L^2 + \alpha^2 \ . \tag{4.63}$$

Since $E > 0$, this equation defines the upper hyperbola in the space of dynamical integrals. The calculation of the symplectic form (4.62) then yields

$$\omega = \frac{dL \wedge dK}{E} \ , \tag{4.64}$$

with $E = \sqrt{K^2 + L^2 + \alpha^2}$. The inversion of this symplectic form provides the Poisson bracket relations of the $o(2,1)$ algebra

$$\{L,\, K\} = E \ , \qquad \{E,\, K\} = L \ , \qquad \{E,\, L\} = -K \ . \tag{4.65}$$

Note the generators E, K, L have an oscillator representation with $E = H + \alpha$, where H is the harmonic oscillator hamiltonian (see the exercise 5 of the lecture 2).

The 3-dimensional case with the hyperboloid (4.45) is discussed in the exercise 15.

Appendix

The $SU(2)$ group is given as the set of 2×2 unitary matrices with the unit determinant. Similarly to $SL(2,\mathbb{R})$ (see Lecture 3), an element $g\in SU(2)$ and its inverse g^{-1} can be written as

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}~, \qquad g^{-1}=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}~, \qquad \text{with} \qquad ad-bc=1~. \tag{A.1}$$

But now the numbers a, b, c, d are complex. The unitarity condition $g^{+}\,g=I$ is equivalent to $g^{+}=g^{-1}$ and from (A.1) one obtains $d=a^*$, $c=-b^*$. As a result, the $SU(2)$ group elements are parameterized by

$$g=\begin{pmatrix} a & b\\ -b^* & a^*\end{pmatrix}~, \qquad \text{with} \qquad |a|^2+|b|^2=1~. \tag{A.2}$$

Introducing the real parameters $a=u_1+iu_2$, $b=u_3+iu_4$, one gets the equation for S^3

$$u_1^2+u_2^2+u_3^2+u_4^2=1~. \tag{A.3}$$

Hence, the $SU(2)$ group is a real 3-dimensional manifold, which is identified with S^3.

The corresponding Lie algebra $su(2)$ is formed by the anti-hermitian 2×2 traceless matrices. Choosing the basis $e_n=-i\sigma_n$, $n=(1,2,3)$, where σ_n are the Pauli matrixes

$$\sigma_1=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}~, \quad \sigma_2=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}~, \quad \sigma_3=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}~, \tag{A.4}$$

one obtains

$$e_m\, e_n=-\delta_{mn}\, I+\epsilon_{lmn}\, e_l~, \tag{A.5}$$

and the commutators

$$[\,e_m,\, e_n\,]=2\epsilon_{lmn}\, e_l~. \tag{A.6}$$

Thus, the $su(2)$ algebra is an Euclidean version of $sl(2,\mathbb{R})$ and the definitions and constructions discussed in Lecture 3 have a natural generalization. Below we give the corresponding list of formulas, which are similar to the $SL(2,\mathbb{R})$ case.

- The normalized trace: $\langle\, M\,\rangle=-1/2\,\mathrm{Tr}(M)$.
- The normalized traces of products of the basis vectors:

$$\langle\, e_m\, e_n\,\rangle=\delta_{mn}~, \qquad \langle\, e_l\, e_m\, e_n\,\rangle=\epsilon_{lmn}~. \tag{A.7}$$

- The maps between $su(2)$ and $\mathbb{R}^3$: $A=A_n\, e_n\in su(2)$, $A\mapsto A_n$, $A_n=\langle\, e_n\, A\rangle$.
- The square of an element $A\in su(2)$: $A^2=-\langle\, A\, A\rangle\, I$.
- The exponential map:

$$e^A=\cos\theta\;I+\sin\theta\;\hat{A}~, \qquad \text{with} \quad \theta=\sqrt{\langle\, A\, A\,\rangle}~, \quad \hat{A}=\frac{A}{\theta}~. \tag{A.8}$$

- The adjoint representation: $\mathrm{Ad}_g(A)\mapsto g\,Ag^{-1}$ and the map from $SU(2)$ to $SO(3)$:

$$g\mapsto O_{mn}=\langle\, e_m\, g\, e_n\, g^{-1}\,\rangle~. \tag{A.9}$$

- The left-right vector fields on $SU(2)$: $\hat{L}_n(g)=e_n\, g$, $\hat{R}_n(g)=g\, e_n$.
- The left-right 1-forms on $SU(2)$: $L_n=\langle e_n\, dg\, g^{-1}\rangle$, $R_n=\langle e_n\, g^{-1}\, dg\rangle$.
- The metric tensor on $SU(2)$:

$$\mathrm{g}_{\mu\nu}=\langle\, g^{-1}\,\partial_\mu\, g\, g^{-1}\,\partial_\nu\, g\rangle~. \tag{A.10}$$

Exercises

1. The space of motions M for the oscillator problem $\ddot{x} + x = 0$, can be written as

$$x(t) = A\,\cos(t + \alpha)\ . \tag{E.1}$$

Calculate the symplectic form on M in terms of the parameters $(A,\,\alpha)$.

2. Show that the parameter p in (4.7) corresponds to the asymptotic *in*-momentum of the Liouville particle and $-p$ is the *out*-momentum, respectively.
Find the *in* and *out* coordinates in terms of p and q.
Constuct the canonical transformation from *in* to *out* variables, show that it has the form

$$(p_{in}, q_{in})\ \mapsto\ (p_{out}, q_{out}) = (-p_{in}, -q_{in} + \beta'(p_{in}))\ , \tag{E.2}$$

and find the function $\beta(p)$ (up to a constant).

3. The Lie derivative of a metric tensor $\mathrm{g}_{\mu\nu}$ in components is given by (see the exercise 2.9)

$$(\mathcal{L}_V \mathrm{g})_{\mu\nu} = V^\sigma \partial_\sigma \mathrm{g}_{\mu\nu} + \mathrm{g}_{\mu\sigma}\partial_\nu V^\sigma + \mathrm{g}_{\sigma\nu}\partial_\mu V^\sigma\ . \tag{E.3}$$

Check that if $(\mathcal{L}_V \mathrm{g})_{\mu\nu} = 0$, then the free particle Lagrangian (4.15) is invariant under the infinitesimal transformations $q^\mu \mapsto q^\mu + \epsilon\, V^\mu(q)$.

4. Check that if $(\mathcal{L}_V \mathrm{g})_{\mu\nu} = 0$ (see the previous exercise), then $C = \mathrm{g}_{\mu\nu}(q)\, V^\mu(q)\dot{q}^\nu$ is a dynamical integral $\dot{C} = 0$.

5. Describe the geodesics on the sphere $x^2 + y^2 + z^2 = R^2$.

6. Check that the metric tensor on the $SU(2)$ group manifold (A.10) coincides with the standard metric on S^3 induced from $\mathbb{R}^4$

$$ds^2 = du_1^2 + du_2^2 + du_3^2 + du_4^2\ . \tag{E.4}$$

7. From (4.27) verify that the $su(2)$ valued matrix $l = \dot{g}g^{-1}$, is also a dynamical integral ($\dot{l} = 0$) and it is related to r by $l = grg^{-1}$.

8. Check that l and r (see the previous exercise) have the same norm $\langle l^2\rangle = \langle r^2\rangle$.

9. Check that the variable $l_n = \langle l\, e_n\rangle$ (see the exercise 7) is given by $l_n = O_{nm}\, r_m$, with $O_{mn} = \langle e_n\, g\, e_m\, g^{-1}\rangle \in SO(3)$.

10. Using the Poisson brackets (4.37), (4.38) and (4.39) check that l_n satisfies the following Poisson bracket relations

$$\{l_n, l_m\} = -2\epsilon_{knm}\, l_k\ , \qquad \{l_n, g\} = e_n\, g\ , \qquad \{l_m, r_n\} = 0\ . \tag{E.5}$$

11. Let us parameterize the $(N+1)$-dimensional hyperboloid (4.40) by

$$Y_0 = R\,\sinh\theta\ , \qquad Y_n = R\,\cosh\theta\ u_n\ , \tag{E.6}$$

where u_n is an unit vector ($u_n u_n = 1$) in $\mathbb{R}^{N+1}$, which defines the N-dimensional sphere. Check that the induced metric is given by

$$ds^2 = R^2 (d\theta)^2 - R^2\,\cosh^2\theta\ (du_n\, du_n)\ . \tag{E.7}$$

12. In the AdS case, the hyperboloid (4.41) can also be parameterized by

$$X_{0'} = R \cosh\rho \sin\theta \ , \qquad X_0 = R \cosh\rho \cos\theta \ , \qquad X_n = R \sinh\rho \; u_n \ , \tag{E.8}$$

where now u_n is an unit vector in $\mathbb{R}^N$. Check that the induced metric in these coordinates is

$$ds^2 = R^2 \cosh^2\rho \,(d\theta)^2 - R^2 \left[(d\rho)^2 + \sinh^2\rho \; (du_n \, du_n)\right] \ . \tag{E.9}$$

13. Using the parametrization (4.42) of the hyperboloid (4.41), form (4.51) one finds the following form of the trajectories

$$X_0 = r(\theta) \cos\theta \ , \qquad X_{0'} = r(\theta) \sin\theta \ , \qquad X_n = \frac{r(\theta)}{E} \left(K_n \sin\theta - L_n \cos\theta\right) \ . \tag{E.10}$$

Check that the function $r(\theta)$ can be written as

$$r(\theta) = \frac{\sqrt{2}\, RE}{\sqrt{2E^2 - [K^2 + L^2 - (K^2 - L^2)\cos 2\theta - K\cdot L \, \sin 2\theta]}} \ , \tag{E.11}$$

where K^2 and L^2 are given by (4.55) and $K \cdot L = K_n L_n$.

Note that eqs. (E.10)-(E.11) provide a parametrization of the trajectories through the dynamical integrals E, K_n, L_n and the evolution coordinate θ.

14. Check that the condition for the choice of time direction (4.48) is invariant under a boost in the (X_0, X_n) plane. One can use the form of the trajectories (4.51) and the inequality (4.60). The invariance of (4.48) with respect to the rotations $SO(2) \times SO(N)$ is obvious.

15. Here we consider the dynamics on the hyperboloid (4.45). In this case there is only one quadratic relation (4.57), and together with the Casimir condition (4.54) one gets two equations

$$E^2 + J_{12}^2 = K_1^2 + K_2^2 + L_1^2 + L_2^2 + \alpha^2 \ , \qquad E\, J_{12} = K_1 L_2 - K_2 L_1 \ . \tag{E.12}$$

Let us introduce 'left' and 'right' variables (E_l, K_l, L_l) and (E_r, K_r, L_r) defined by

$$\begin{aligned} 2E_l &= E + J_{12} \ , \qquad & 2K_l &= K_1 + L_2 \ , \qquad & 2L_l &= L_1 - K_2 \ ; \\ 2E_r &= E - J_{12} \ , \qquad & 2K_r &= K_1 - L_2 \ , \qquad & 2L_r &= L_1 + K_2 \ . \end{aligned} \tag{E.13}$$

Using the algebra (4.61), check that the left variables have zero Poisson brackets with the right variables and they both satisfy the Poisson bracket relations of the $o(2,1)$ algebra (4.65).

Check also that the two conditions of (E.12) are equivalent to

$$E_l^2 = K_l^2 + L_l^2 + (\alpha/2)^2 \ , \qquad E_r^2 = K_r^2 + L_r^2 + (\alpha/2)^2 \ . \tag{E.14}$$

These equations define two hyperbolas (4.63) with the lowest values of E_l and E_r equal to $\alpha/2$.

Find the oscillator representation of the symmetry generators (E.13), using the above mentioned splitting $o(2,2) = o(2,1) \oplus o(2,1)$ and the result of the exercise 5 of the lecture 2.

Express the $o(2,2)$ generators E, J_{12}, K_n, L_n $(n = 1,2)$ in terms of the 'creation-annihilation' variables defined by

$$a_1 = \frac{a_l + a_r}{\sqrt{2}} \ , \qquad a_2 = i\, \frac{a_l - a_r}{\sqrt{2}} \ , \tag{E.15}$$

where a_l and a_r are the 'annihilation' variables for the left $(E_l,\ K_l,\ L_l)$ and the right $(E_r,\ K_r,\ L_r)$ generators, respectively.

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